Eliminating redundant columns from column generation subproblems using classical Benders' cuts

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SCIP workshop 2018 · Aachen · 08/03/2018

Dantzig-Wolfe reformulation for IPs

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \\ & Dx \geq d \\ & x \in \mathbb{Z}^n_{\geq 0} \end{aligned}$$

- original problem

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$$\{x \in \mathbb{Z}^n_{\geq 0} : Dx \geq d\} = \bigcup_p \{x^p\}$$

- substitute x-variables with λ-variables

$$\begin{aligned} \textstyle\sum_p x^p \lambda_p &= x \\ \textstyle\sum_p \lambda_p &= 1 \\ \lambda &\in \{0,1\}^q \end{aligned}$$

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- master IP
- solve master LP with col.gen.
- col.gen. subproblem

$$\begin{aligned} \min\ & \texttt{redcost}(x) \\ \text{s.t. } & Dx \geq d \\ & x \in \mathbb{Z}^n_{\geq 0} \end{aligned}$$

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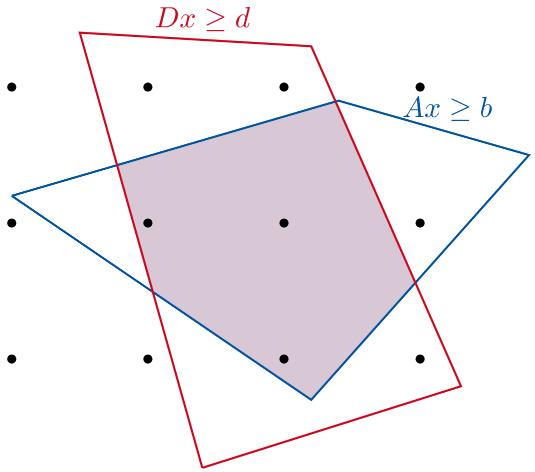

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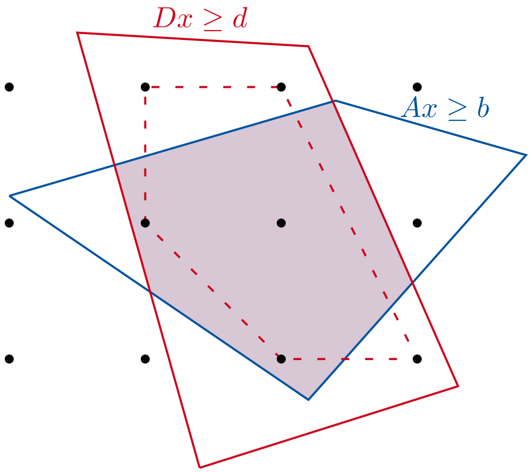

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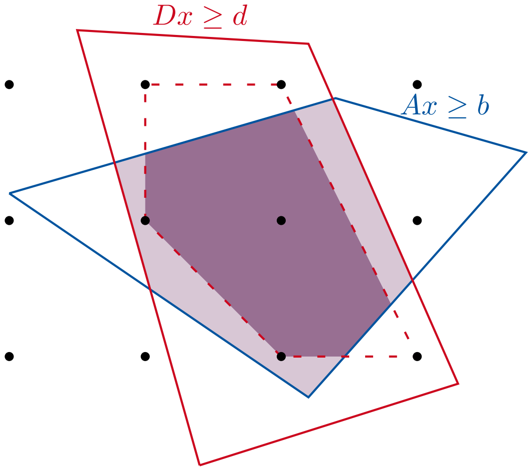

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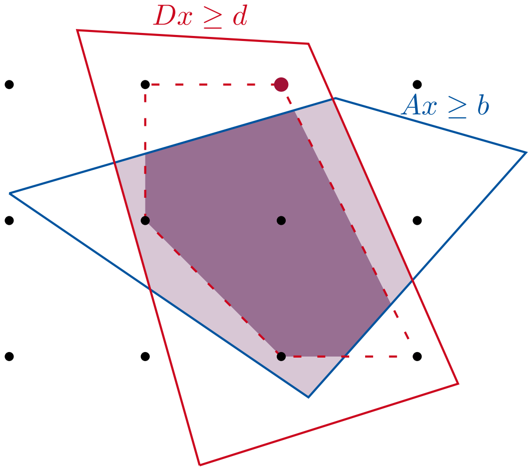

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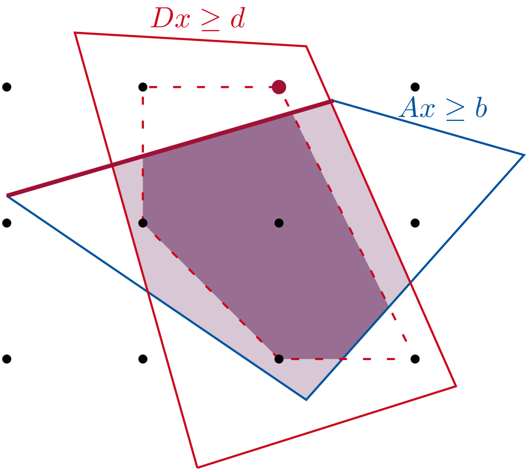

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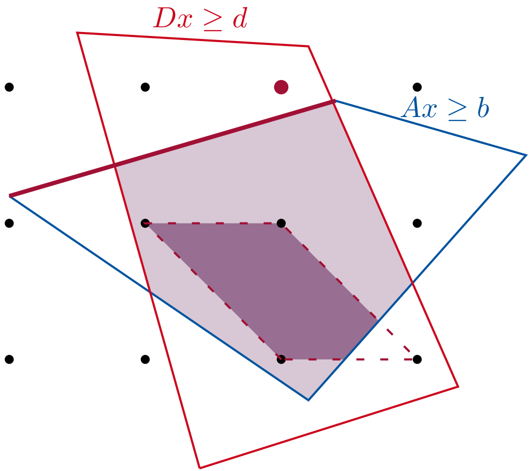

Literature

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  Vanderbeck and Savelsbergh (2006)

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- refine subproblem to eliminate (some) redundant columns
  - until now: only domain propagation for tighter variable bounds in subproblems Vanderbeck and Savelsbergh (2006); Gamrath and Lübbecke (2010)
  - this talk: add inequalities/cuts to subproblems

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  - this talk: add inequalities/cuts to subproblems
- column is *strongly redundant* if it is not part of any optimal solution to the master IP

Redundant columns

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\begin{array}{rlclcll}
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& & & D_2 x^2 & \geq & d_2 & \\
& & & x^k & \in & \mathbb{Z}_{\geq 0}^{n_k} & \forall k \in \{1, 2\}
\end{array}
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- set F of feasible solutions
- set F^k of feasible solution to subproblem $k \in \{1, 2\}$

$$F^k = \{x^k \in \mathbb{Z}_{\geq 0}^{n_k} : D_k x^k \geq d_k\}$$

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- how do we check if a column $\bar{x}^1 \in F^1$ is strongly redundant?

$\rightarrow$ check if $\exists \bar{x}^2 \in F^2$ with

1. $(\bar{x}^1, \bar{x}^2) \in F$
2. $c_1^T \bar{x}^1 + c_2^T \bar{x}^2 \leq z^*$

Redundant columns

- check strong redundancy of $\bar{x}^1$ with feasibility problem:

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→ upper bound z^{UB} instead of z^*

→ relax integrality of x^2

→ classical Benders' feasibility cuts to refine subproblem

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Subproblem refining inequalities

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$$\begin{array}{lrcl} \max & \texttt{dualobj}(\pi, \bar{x}^1) & & \\ \text{s. t.} & (*)\ \pi & \leq & 0 \\ & \underbrace{\pi_A, \pi_D, -\pi_c}_{\pi} & \geq & 0 \end{array}$$

- dual polyhedron independent of $\bar{x}^1$

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- if primal infeasible, $\exists$ Farkas proof (dual ray) $\bar{\pi}$ with:

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$$\texttt{dualobj}(\bar{\pi}, \bar{x}^1) > 0$$

$\rightarrow$ valid inequality for all x^1, not strongly redundant

$$\texttt{dualobj}(\bar{\pi}, x^1) \leq 0$$

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Subproblem refining inequalities

$$\overbrace{\pi_A^T(b - A_1\bar{x}^1) + \pi_D^T d_2 + \pi_c \cdot (z^{UB} - c_1^T \bar{x}^1)}^{\texttt{dualobj}(\pi, x^1)} \leq 0$$

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optimality subproblem cut

- alternative way without objective constraints:
  use dual instead of Farkas values, maximizing violation

Elimination of redundant columns

- pricing iteration with redundancy check:
  1. solve col.gen. subproblems
  2. for each subproblem solution
     - check redundancy with LP
     - if redundant, add subproblem cut
  3. if all found columns are redundant $\rightarrow$ 1

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\+ potentially stronger dual bound with master LP

\+ possibly "better" columns for the master IP

– solve LP for each subproblem solution

– subproblems can become more difficult to solve

Implementation

- implemention in generic BP&C solver GCG based on SCIP
- separator in the master problem
  - callbacks `SEPAEXECLP` and `SEPAEXECSOL` reduce the domain

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  - cuts are by default only separated in the pricing loop

  $\rightarrow$ public methods for separation called in pricer

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- add parameter forcing SCIP to allways compute dual solution

  ```
  SCIP_CALL( SCIPaddBoolParam(..., "misc/alwaysgetduals", ...) );
  ```

Computational tests

- other separators and stabilization disabled for computations
- redundancy check in all pricing iterations at root node

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- information from original LP relaxation is used
- weak original LP relaxations

Other instances

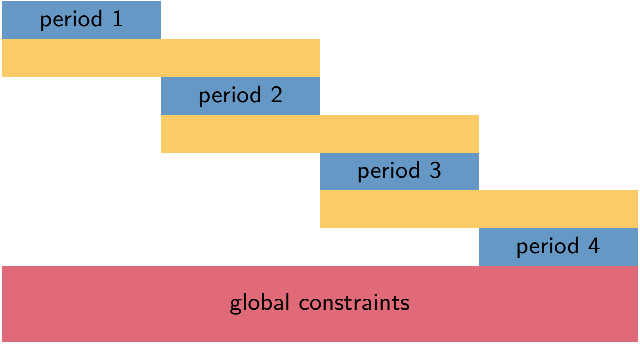
- capacitated lot sizing problems
  Tempelmeier and Derstroff (1996); Trigeiro et al. (1989)
  - period decomposition
    Pimentel et al. (2010); de Araujo et al. (2015)
  - horizon decompositions
    Fragkos et al. (2016)

- linearized thermal unit commitment instances
  Frangioni et al. (2009)
  - period, horizon decompositions
    Kim et al. (2017)

- subset of MIPLIB instances Bergner et al. (2015)
  - automatic detection with GCG
  - max. white score

Time decompositions

- "time-dependent" structure in lot sizing and unit commitment:



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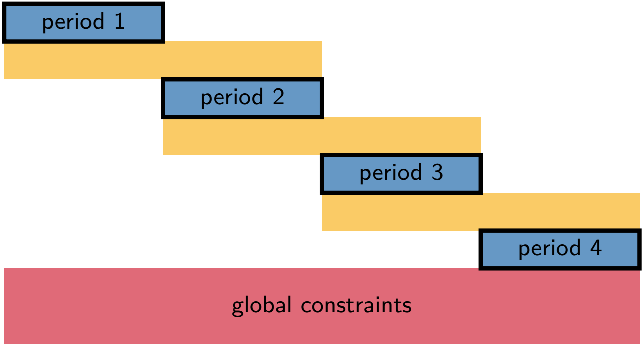

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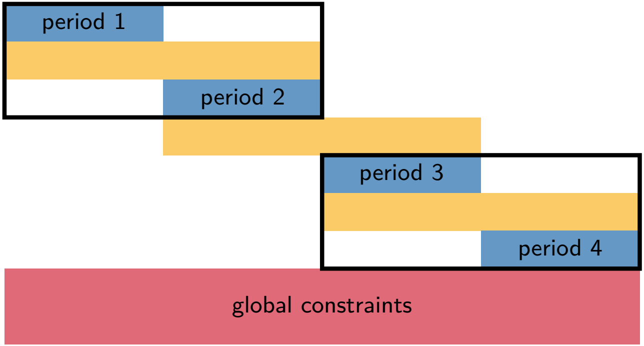


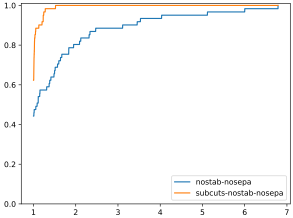
- horizon decomposition with horizons of 2 periods

Overview

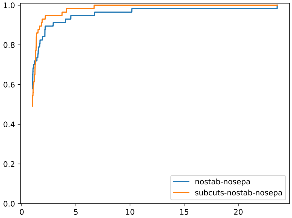
	overall		subcuts		default	
	ninstances	naffected	nsolved	gmeantime	nsolved	gmeantime
ls-derstroff-period	98	75	61	174.48	62	201.55
ls-derstroff-horizon2	98	69	62	117.17	57	157.80
ls-trigeiro-horizon2	69	61	0	3600.00	0	3600.00
ls-trigeiro-horizon4	69	65	9	3079.39	9	3320.52
ls-trigeiro-horizon8	69	50	67	158.08	66	150.11
uc-period	42	30	7	2161.37	7	2067.80
uc-horizon2	42	42	6	2273.87	5	2473.20
uc-horizon4	42	42	6	2119.09	6	2336.34
uc-horizon8	42	31	6	1873.31	6	2200.36
uc-horizon12	42	29	8	1493.45	8	1585.63
miplib-maxwhite	39	13	7	1355.43	7	1343.56

- time to solve root usually increases with subcuts
- behavior of number of col.gen. iterations unpredictable

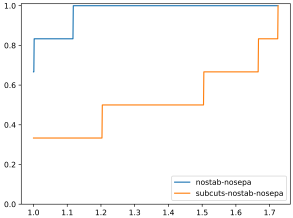
Lot sizing - performance profiles



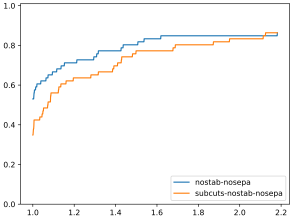
(a) Derstroff period



(b) Derstroff horizon 2



(c) Trigeiro horizon 4



(d) Trigeiro horizon 8

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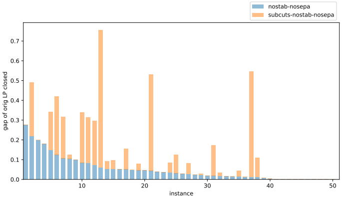
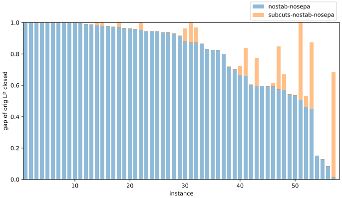
Lot sizing - root gap closed



(e) Derstroff period



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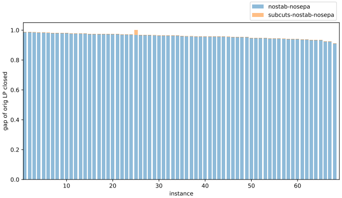
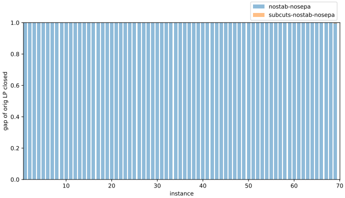


(g) Trigeiro horizon 4



(h) Trigeiro horizon 8

Lot sizing

- one subproblem for each period
- column $\leftrightarrow$ production plan for period t
- periods are linked by demand/balance constraints

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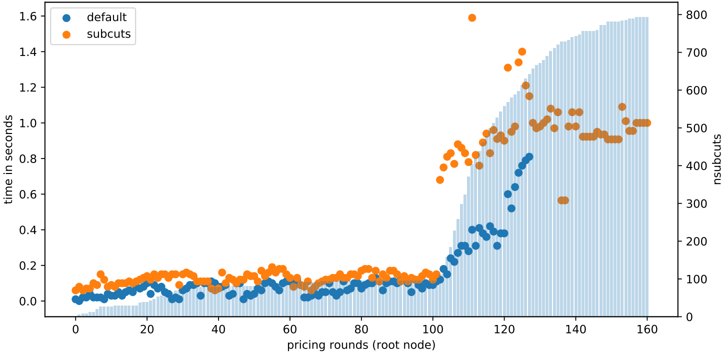
$$y_{it} = \begin{cases} 1, \text{if product } i \text{ is produced in period } t \\ 0, \text{otherwise} \end{cases}$$

- with setup times and costs

- subproblem cuts for period t and subset I' of products

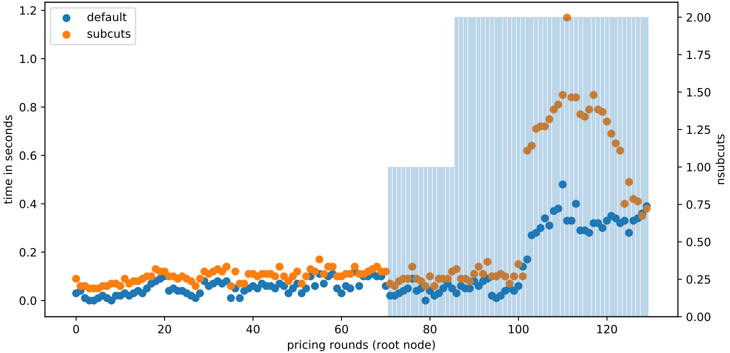
$$\sum_{i \in I'} y_{it} \geq 1$$

Effect on pricing



(i) K8021551-period (Derstroff)

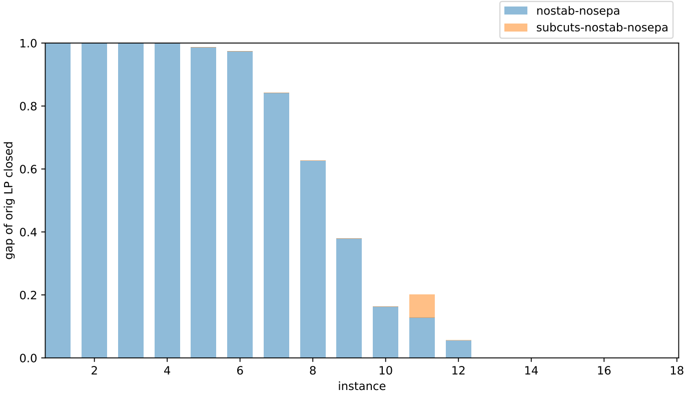
Effect on pricing II



(j) G0065252-period (Derstroff)

MIPLIB instances

- automatic detection (max. white)



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Future work

- control of pricing difficulty?
- more problem classes with "time-dependent" structure?
- other decompositions for MIPLIB instances
- effect on price-and-branch heuristic

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