Investigating Mixed-Integer Hulls using a MIP-Solver

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Joint work with

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Outline

Problems in Question

We consider mixed-integer programs with rational data:

$$\begin{aligned} \max\ & \langle c, x \rangle \\ \text{s.t.}\ & Ax \leq b \\ & Cx = d \\ & x_i \in \mathbb{Z} \qquad \forall i \in I \subseteq [n] \end{aligned}$$

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Denote by $R = \{x \in \mathbb{R}^n : Ax \leq b,\ Cx = d\}$ the relaxation polyhedron and by $P = \text{conv.hull}\,\{x \in R : x_i \in \mathbb{Z}\ \forall i \in I\}$ the mixed-integer hull.

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Facts

- P is a polyhedron again.
- For most (e.g., NP-hard) problems, P has many facets.
- Nevertheless, MIP-solvers are really fast these days.

Another Fact

The time a MIP-solver needs for solving depends on the strength of the relaxation, i.e., how well P is approximated by R.

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- Problem-specific inequalities: Problem-dependent

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Goals of Polyhedral Combinatorics

Given a MIP-model for a problem,

- find inequalities valid for P (but not for R),
- develop algorithms (exact or heuristics) to separate these inequalities if there are too many,
- determine the dimension of P, i.e., find valid equations,
- and prove if/when the inequalities define facets of P.

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There are quite a few tools (`PORTA`, `Polymake`, `azove`) and several algorithms:

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- The Double-Description method
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- Pyramid decomposition (mixture of beneath-and-beyond and Fourier-Motzkin)

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Step 3: Generalize Inequalities

There's only one main tool here: The mathematician.

Memory and Time

The dominant of the cut polytope (corresponds to MinCut problem) has among others a facet per disjoint union of cycles joined together by any spanning tree!

Alevras ('99) enumerated the facets for this polyhedron for the complete graph on 8 nodes, including 2 billions of the above type.

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Specific Objective Functions

Which are the facets useful when optimizing specific objective functions?

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Goal of this work:

Use MIP-solvers to determine facets!

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We call a MIP-oracle stable if all reported solutions are correct after replacing the floating-point numbers by the best approximation w.r.t. to given tolerances.

Note: A rational number p/q with $p, q \in \mathbb{Z}$ is a best approximation for $x \in \mathbb{R}$ if $|x - p/q| \leq |x - p'/q'|$ holds for all $p', q' \in \mathbb{Z}$ with $q' \leq q$.

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Question: How does such an oracle look like in practice?
Answer: `SCIP*` and some luck with the numerics!

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Algorithm:

1. Maintain known equations $Cx = d$ and points $x_1, x_2, \dots, x_\ell \subseteq P$.
2. Let A be matrix C with additional rows $x_i - x_1$ for $i = 2, \dots, \ell$.

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      3. Otherwise, $\langle c, x \rangle = \langle c, x_1 \rangle$ defines a valid equation for P and hence can be added to $Cx = d$. Add row c to A.

Computing the Affine Hull (cont.)

Remark

- *We can initialize* $Cx = g$ *with equations from LP relaxation.*
- *Optimizing in unit directions is cheap but yields few points.*
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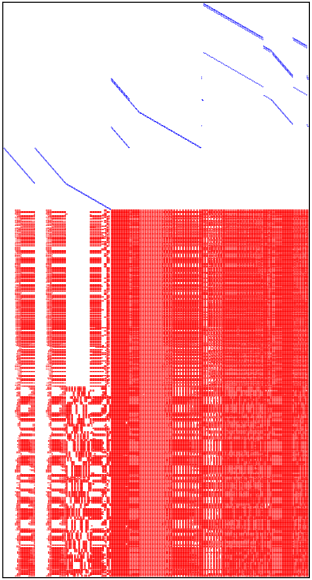
Application

Instead of the rows $x_i - x_1$ for $i = 2, \dots, \ell$, we use $x_i - x_j$ for all minimum spanning tree edges $\{i, j\}$ where the weights are the number of nonzeros of $x_i - x_j$. This leads to a sparse (but equivalent) matrix A.

Example for Input Matrix



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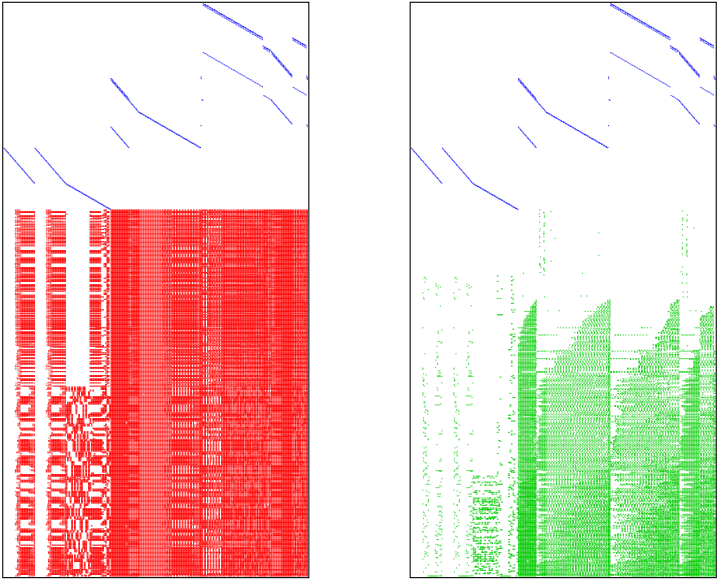

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Let $P \subseteq \mathbb{R}^n$ be a polyhedron with $\mathbb{O}_n$ in its relative interior. Then the set

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Since the polar dual has a lineality space if P is not full-dimensional, let

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Proposition (see Schrijver '86)

- $(P^*)^* = P$
- *x is a point in (vertex of) P if and only if $\langle x, y \rangle \leq 1$ defines an inequality for (facet of) P^*.*
- *P^* is bounded if and only if P contains $\mathbb{O}$ in its relative interior.*

Suppose $\mathbb{O}$ is in the relative interior of P.

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We are given a point $\hat{x} \in R \setminus P$. Find a facet $\langle y, x \rangle \leq 1$ of P cutting off $\hat{x}$!

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Very related work: Target Cuts by Buchheim, Liers & Oswald, '08

Motivation

- Readability of produced equations & facets!
- $\langle a, x \rangle \leq \beta$ with $a \in \mathbb{Z}^n$ and $\beta \in \mathbb{Z}$ where $(a^\intercal, \beta)$ is a primitive vector.

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Input are two linearly independent vectors $u, v \in \mathbb{Z}^n$.
Find $\lambda, \mu \in \mathbb{Q}$ *with* $\lambda \neq 0$ *and* $\lambda u + \mu v \in \mathbb{Z}^n$ *minimizing* $|\lambda u + \mu v|_1$.

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Lemma

By computing the Hermite-Normal-Form of $\begin{pmatrix} u^\intercal \\ v^\intercal \end{pmatrix} \in \mathbb{Z}^{2 \times n}$, *we can replace* u, v *by a lattice basis* u', v' *of* $\text{lin}\,(u, v) \cap \mathbb{Z}^n$ ...

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By computing the Hermite-Normal-Form of $\begin{pmatrix} u^\intercal \\ v^\intercal \end{pmatrix} \in \mathbb{Z}^{2 \times n}$, *we can replace* u, v *by a lattice basis* u', v' *of* $\mathrm{lin}\,(u, v) \cap \mathbb{Z}^n$ ...

...*such that* $\lambda u' + \mu v'$ *is a multiple of* u *if and only if* $\lambda = 0$.

Problem (1-Norm Minimization in Dimension Two)

Input are two linearly independent vectors $u, v \in \mathbb{Z}^n$.
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Problem (Restricted 1-Norm Minimization in Dimension Two)

Given a lattice basis $u, v \in \mathbb{Z}^n$, *find* $\lambda, \mu \in \mathbb{Z}$ *with* $\lambda > 0$ *minimizing* $|\lambda u + \mu v|_1$.

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Then the sign-pattern of $\lambda u + \mu v$ *is constant over all multiplier pairs* λ, μ *with with* $\lambda > 0$ *for which* λ / μ *is in any fixed interval* $[q_{j-1}, q_j]$.

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Final Step: In such an interval the 1-norm is linear and integer programming in dimension 2 can be solved efficiently (see Eisenbrand & Laue, '03).

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Heuristic:

1. Let $w := u$.
2. **For all** $i \in [n]$ **with** $v_i \neq 0$ **do:**
   1. Let $w^{(i)} := v_i \cdot u - u_i \cdot v$.
   2. Divide $w^{(i)}$ by the g.c.d. of its entries.
   3. If $|w^{(i)}|_1 < |w|_1$, replace w by $w^{(i)}$.
3. Return w.

Problems to be tackled by help of MIP-solvers:

- Compute all valid equations / the dimension d.
- Find facets useful when optimizing certain objective functions, together with d independent points proving that it is a facet.
- Compute the dimension of other faces, e.g. the optimal face.
- Compute the conflict graph for 0/1-variables.

Questions?

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To be done:

- Finish basic code :-)
- Improve convergence for cutting plane procedure.
- Carry out computational study.
- Find some nice facets for interesting polytopes.