Outer-Point Generation –
A Decomposition Algorithm for solving MINLPs

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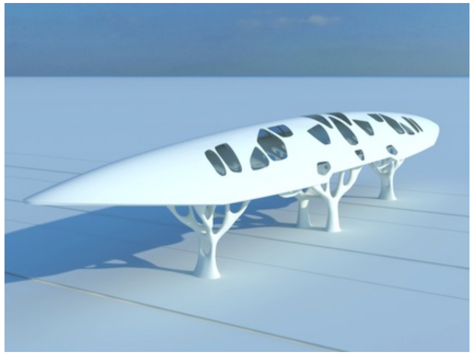
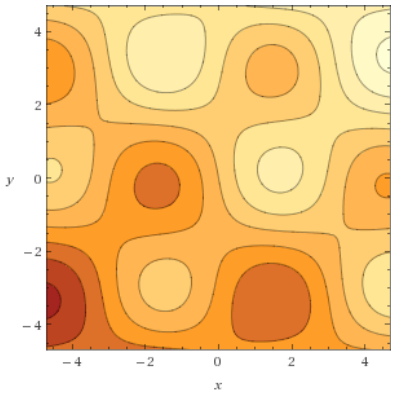
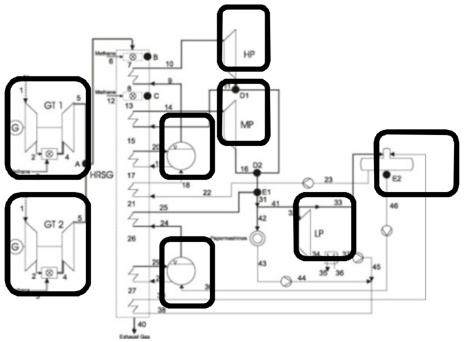
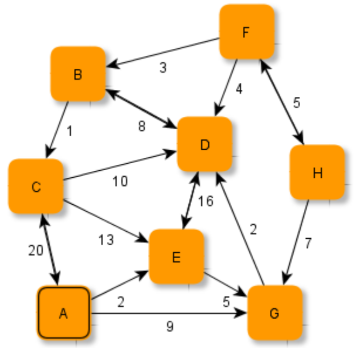








SCIP Workshop, Berlin, 1.10.2014

Main objectives of this talk

1. Present a **new decomposition method,** called **Outer-Point Generation** combining MINLP and Column Generation

2. Discuss **ideas** how this method can be **implemented in SCIP**

Agenda

Motivation: Path Based Column Generation

MINLP Outer-Point Generation

Implementation in SCIP

Next Steps

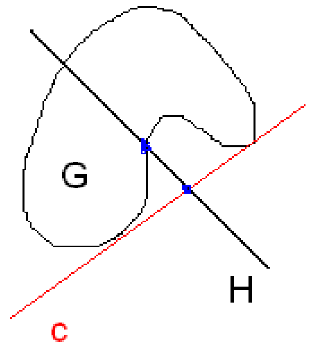
Classical decomposition methods are efficient, if the duality gap is small

Quasi-separable reformulation and **convex relaxation**

optimization problem (P): $c^* := \min\{c(x) : x \in G \cap H\}$

convex relaxation (C): $c_{\text{relax}} := \min\{c(x) : x \in \text{conv}(G) \cap H\}$

duality gap: $\text{gap} := c^* - c_{\text{relax}}$



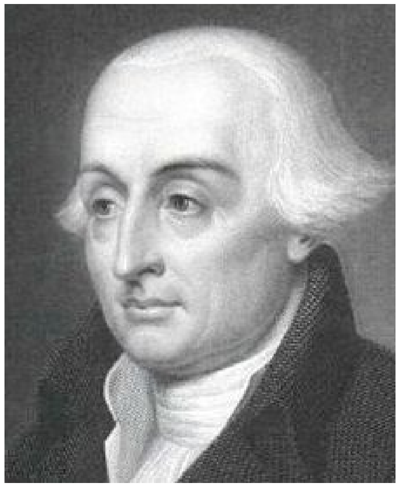
duality gap

Four possibilities for solving the convex relaxation (C)

1) **Lagrangian Decomposition** (extreme points of G)
2) **Column Generation** (Dantzig-Wolfe Decomp, **inner points of G**)
3) **Outer Approximation** (cutting planes of G)
4) **Frank-Wolfe Algorithm** (Linear approximation of the objective)

If **gap is small,** then solution of (C) is a **good starting point** for (P)

Planning & Control: **gap** typically **small**
MINLP: **gap** typically **large**



Joseph-Louis Lagrange 1736 -1813

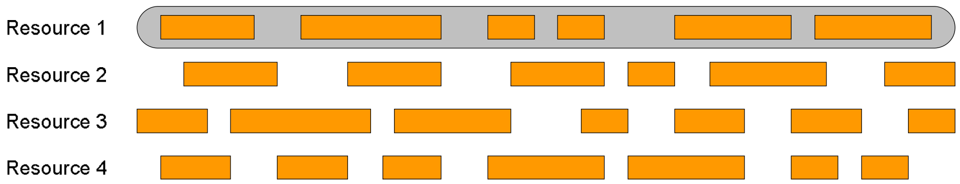
xOPT is a parallel path-based column generation framework for crew and aircraft scheduling developed at Lufthansa Systems

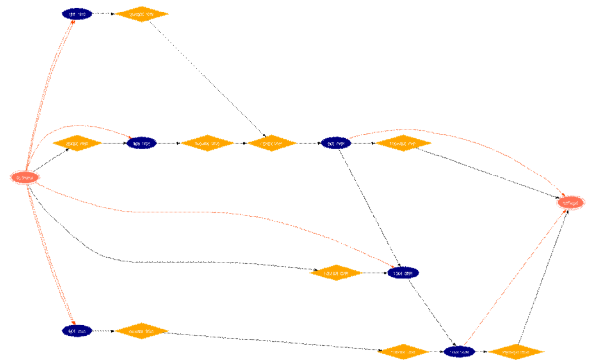
Crew rostering problem:

- variables: **772.927.392**
- (pricing) sub-problems: **3.012**
- coupling constraints: **~500.000**

This problem cannot be solved with a generic optimization tool in reasonable time

The xOPT column generation approach for weakly nonlinear network problems





Constraint Shortest Path pricing problems

$$\text{Min } L_k(p,\pi) \qquad \text{CSP}$$
$$h_k(p) \leq 0 \quad \text{(nonlinear constraints)}$$
$$p \in P_k \quad \text{(network constraints)}$$

LP/IP master problem

$$\text{Min } c(x,s) \qquad \text{LP/IP}$$
$$Ax = b+s \quad \text{(linear constraints)}$$
$$x \in X,\ s \in S$$

CSP-Solver:

- **Label-based dynamic programming** algorithm with relaxed dominance
- Dynamic network filtering and expansion

LP-Solver

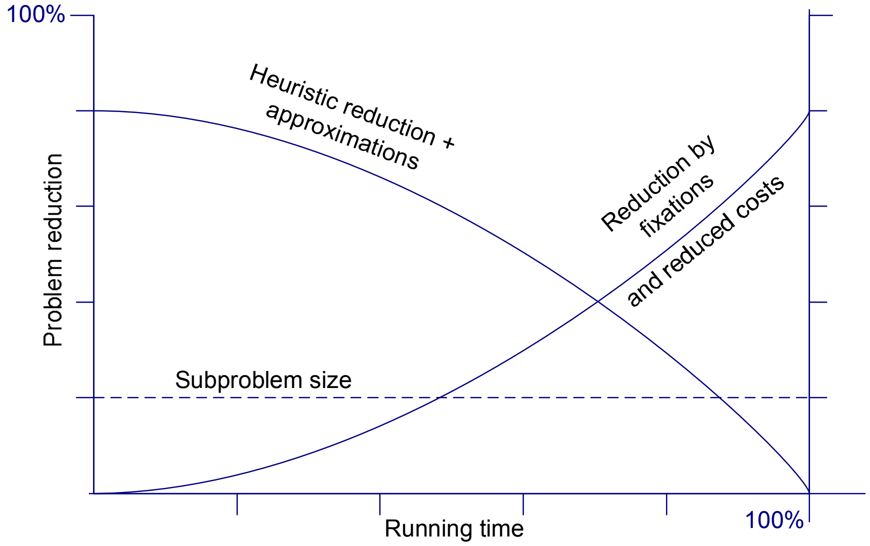
- **Inexact bundle method**
- Interior point start heuristic

IP-Solver:

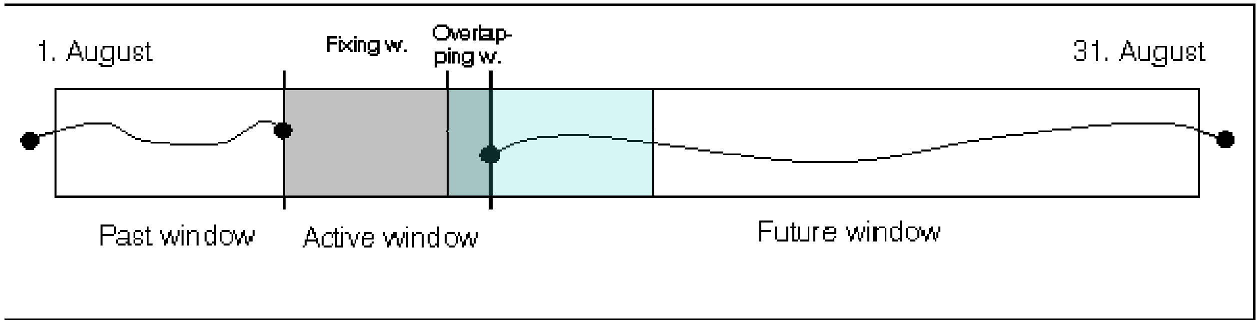
- Column/constraint fixing by **rapid branching** based on
- **perturbation heuristic**

The dynamic Reduce and Generate approach keeps the size of sub-problems manageable

- **Search-space reduction** by a (heuristic) **active set strategy** regarding **columns, constraints, nodes** and **arcs**
- Few main iterations, e.g. 20



- Step-wise fixing by **moving a time-window** over the optimization period

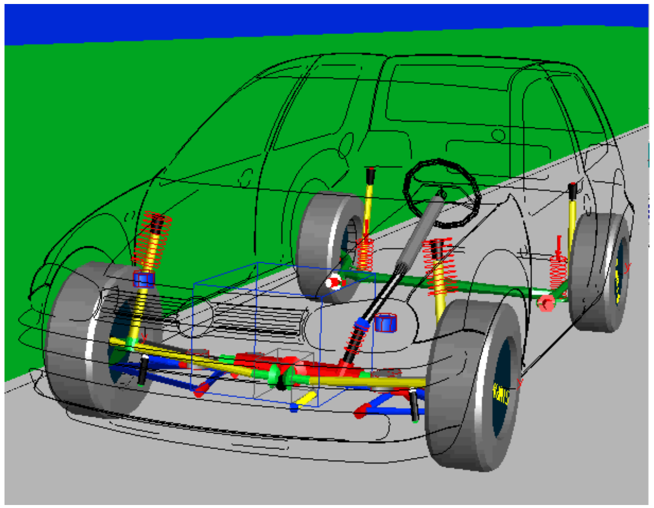


Agenda

Goal: Build a new point-based decomposition method combining MINLP and CG

- The new method shall be used for **industry optimization problems**, which are
  - **sparse (-> decomposition)**
  - strongly **nonlinear, nonconvex, many variables**
  - **simulation-based,** i.e. no algebraic formulation available
- **Currently solved by heuristics** (e.g. evolutionary algorithms)

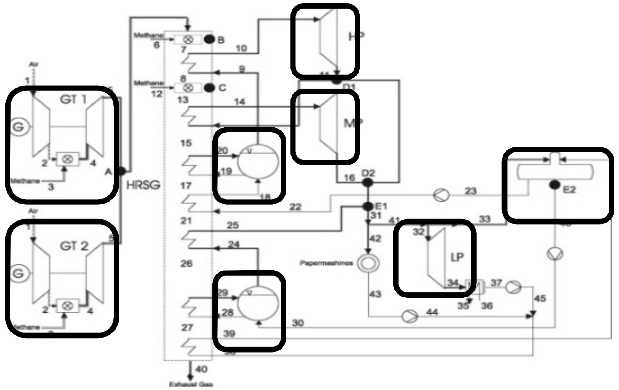
Design Optimization



Quelle: http://www.simpack.com

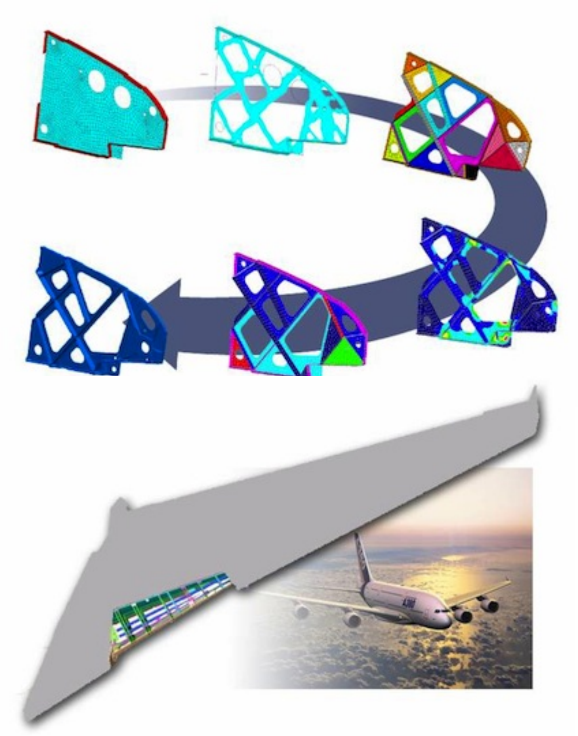
Planning

Energy conversion system (TU-Berlin)

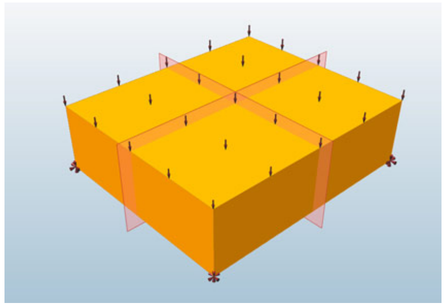


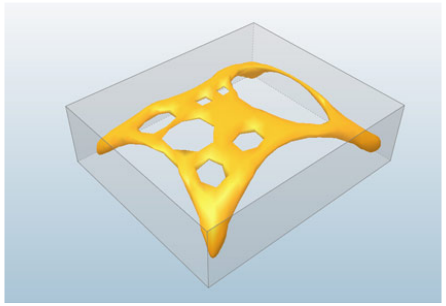
Multidisciplinary Optimization (MDO)

Decomposition of structural optimization problems



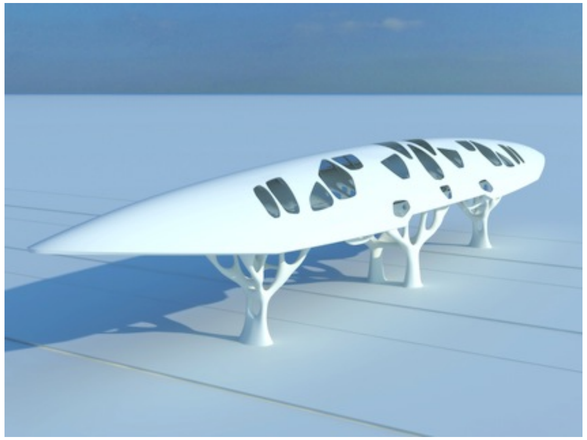
Airbus A380, structural optimization of a wing's rib





Design Space subjected to topology optimization

Examples from Altair/Hyperworks
http://insider.altairhyperworks.com/intuitive-technology-foraec/



Pegasus bridge, final re-design

Decomposition of sparse optimization problems

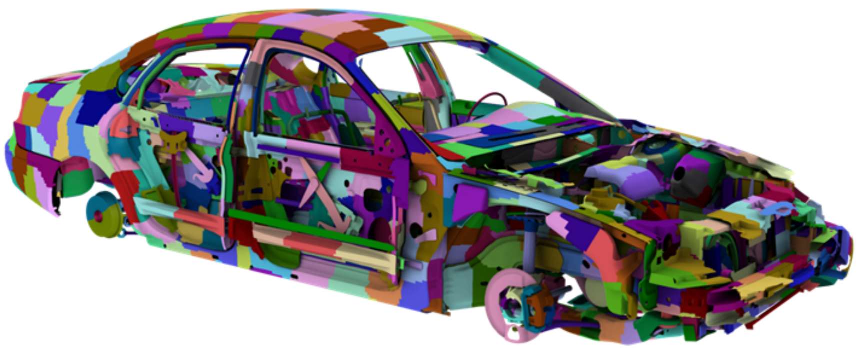
Quasi-separable reformulation by:

- partition problem into **sub-problems** (**domain decomposition**)
- connect sub-problems by **coupling constraints: Ax = 0**

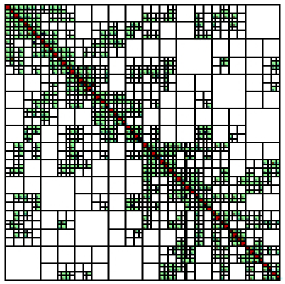
(P) $\min c(x), \qquad Ax = 0, \quad (x_k, y_k) \in G_k, \quad k \in K$

$$G_k := \{x \in X_k : g_k(x) \leq 0\}$$

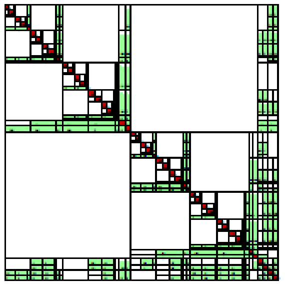
X_k is a mixed integer interval



http://industry.it4i.cz/en/research/basic-research/feti-based-domain-decomposition-methods/



sparse-structure

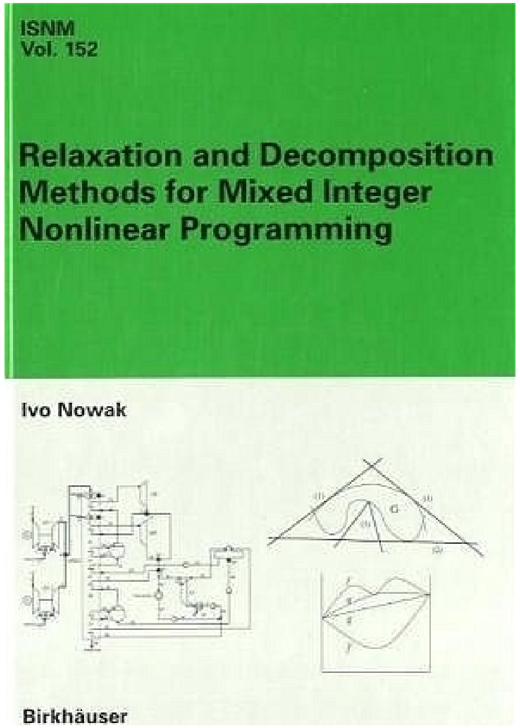


block-structure

(e.g. of a FEM-system)

Preliminary experiments with the MINLP-solver LaGO 2004

- Preliminary implementation of **Lagrangian Decomposition (LD)** for **MINLP** within **LaGO** together with Stefan Vigerske
- **Experiments of LD** with QQPs and MINLPlib showed:
- **LD was not competitive to branch-and-cut because:**
  - MINLPs have typically a **large duality gap.**
  - No special method for **efficiently solving MINLP sub-problems** was used.



What are the key-properties of the CG approach?

How should they be adapted for solving MINLPs?

1. No huge branch-bound trees

CG: **rapid branching** (works if the **duality gap is small**)

MINLP: **Claim 1: Use a nonconvex master problem**

2. Efficient sub-problem solving

CG: (1) **highly tuned algorithm** CSP/DP for enumerating all reduced cost solutions

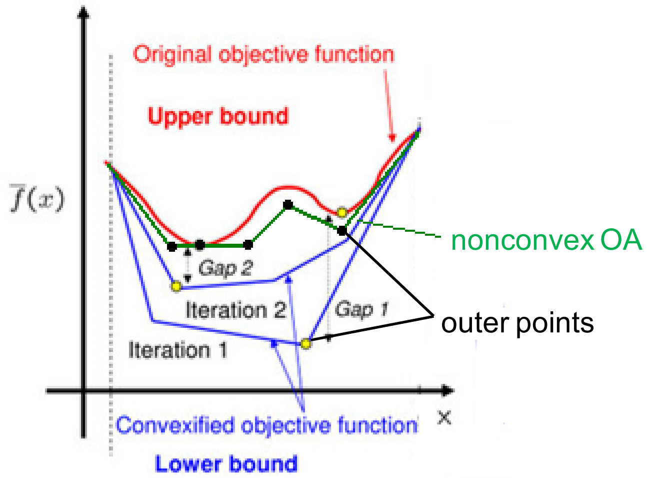
(2) dynamic **search space reduction**

MINLP : **Claim 2: Fast reduction of the approximation error**

The new Outer-Point Generation (OG) approach

Claim 1: Use a nonconvex master problem

- **Idea:**
  - Define the master problem by a nonconvex **point-based Outer Approximation (OA)** based on **Piecewise Linear Envelopes**
  - Improve the OA by **generating** tighter **outer-points (vertices)** of the **OA**



Claim 2: Fast reduction of the approximation error

- **Idea :**
  - Use a **two-phase approach** for reducing the OA-error:

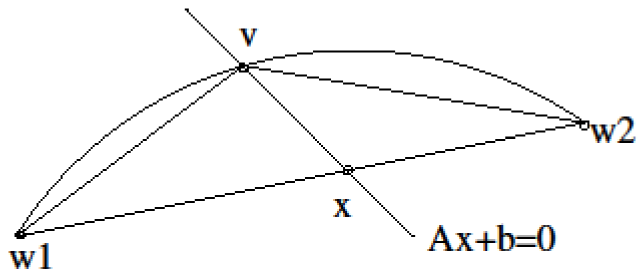
    (1) Column generation

    (2) Sequential approximation: (OA-refinement)

    projection of **ε-OA-solutions** onto the feasible set

v: projected solution

x: OA-solution



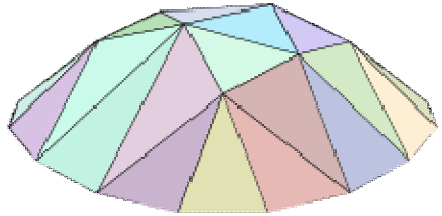
OA refinement

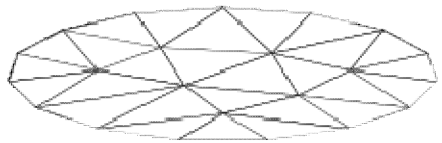
conv {w1,w2} → conv {w1,v} + conv {v,w2}

Remarks

Why OG and not Branch-Cut-And-Price?

- **Cut-based OA is improved by branching** (which we want to avoid) (for integer variables)
- **Point-based OA can be improved without branching** (for ‘nonlinear‘ variables)

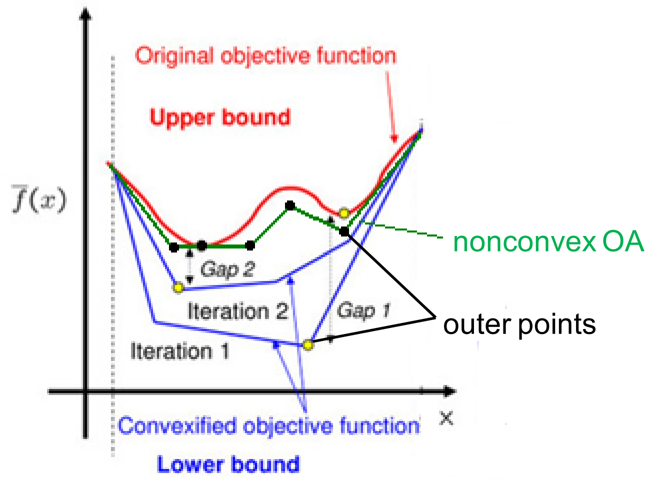




Active set strategy for reducing points (and cuts) **is neccessary**

Derivative-Free Black-Box Optimization

- Sequential approximation of **surrogate models** of black-box functions
- **Refine** surrogate models **at points generated by the OG-algorithm**



Initialization: Outer-Approximation by Piecewise Linear Envelopes

Piecewise linear Outer-Approximation:

$$\text{(OA)} \quad \min c(x), \qquad Ax = 0, \quad (x_k, y_k) \in \hat{G}_k, \quad k \in K$$

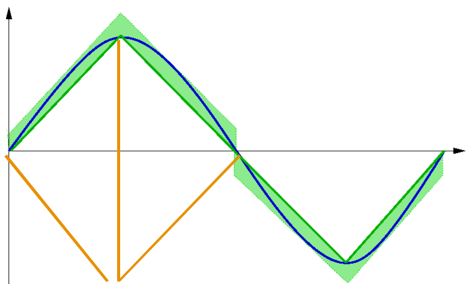
$$\hat{G}_k := \bigcup_{j \in J_k} \operatorname{conv}(V_{kj})$$

x: primary (coupling) variables

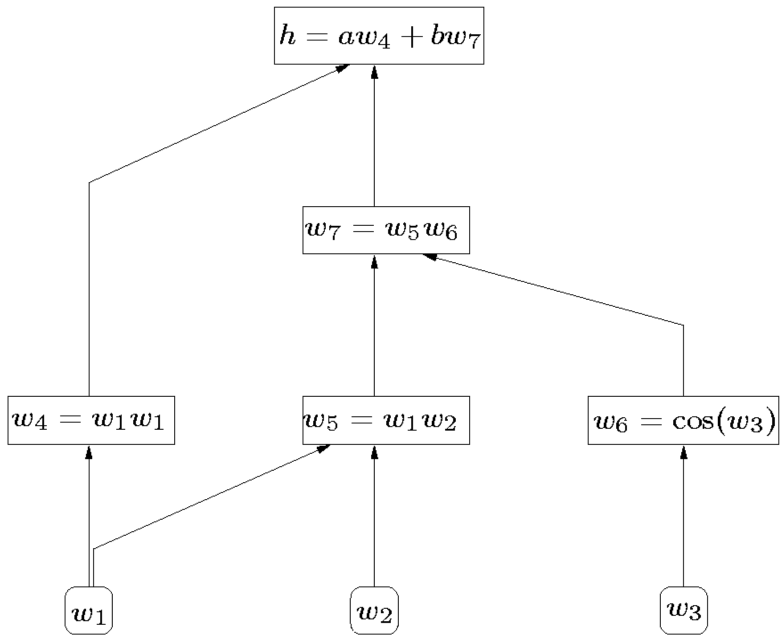
y: secondary (nonlinear) variables

Branch-and-Refine approach of S. Leyffer, A. Sartenaer, and E. Wanufelle, 2008:

- Decompose nonlinear functions into **bilinear** and **univariate** expressions using **expression trees**
- Replace **bilinear** and **univariate** expressions by **PLEs (Piecewise Linear Envelopes)** using **outer-points**



2 PLEs with 6 outer-points

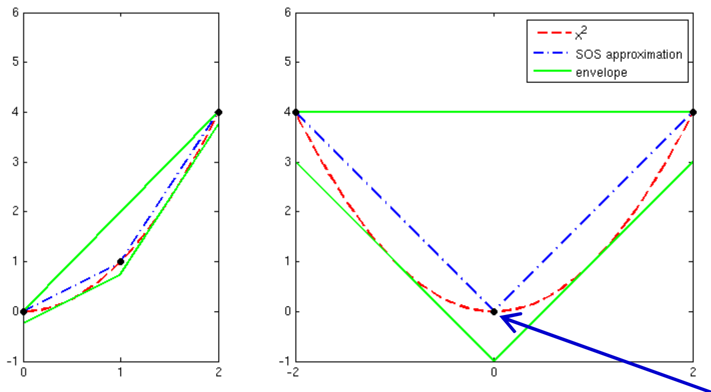


g(x)=ax_1^2 + bx_1x_2cos(x_3)

Refinement of Piecewise Linear Envelopes (PLEs)

Local refinement of a univariate PLE

at an interpolation point

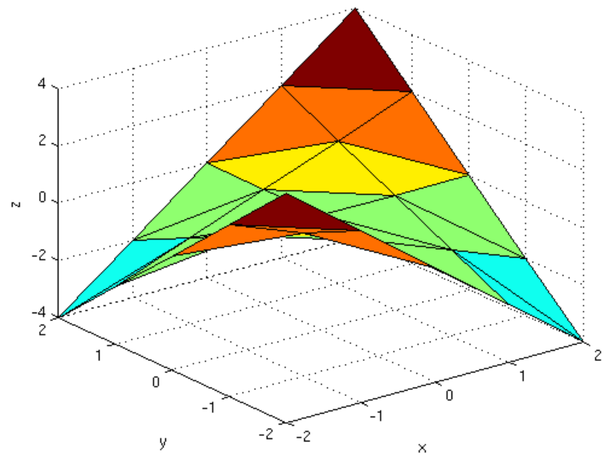


refined PLE

6 outer points of an initial PLE

interpolation point

Uniform refinement of a PLE of a bilinear function



- Outer-points of a PLE are chosen such that the **approximation error is minimized**

- **OA-error** depends on size of linear pieces of the PLEs
- **Continuity of g**: If the OA-error is small enough, then the **solution of (P)** is in the **neighborhood** of an **ε-OA-solution** (-> ε-solving the OA gives the solution)

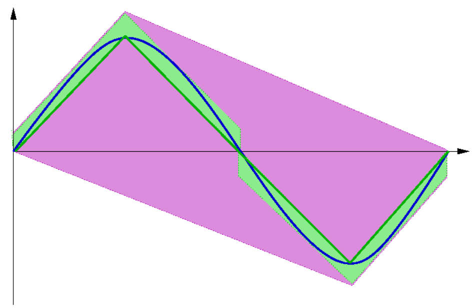
Phase 1: Column Generation

Goal: Reduce the OA-error up to the duality gap (optional)

LP – master problem in the space of **primary variables x**

(LP-MP) $\min c(x), \quad Ax = 0, \quad x_k \in \text{conv}(W_k), \quad k \in K$

$$W_k := \{\hat{x}_{ki} : i \in I_k\} \quad (\hat{x}_{ki}, \hat{y}_{ki}) \in V_k$$



Column generation

1. Compute a **dual solution** of (LP-MP)
2. **Compute ε-solutions** (columns) of the MINLP **pricing sub-problems** regarding a reduced cost direction
3. **Local refinement of PLEs** in the neighborhood of sub-problem solutions

Phase 2: Sequential Approximation

Goal: Reduce the OA-error up to ε by **improving Lower and Upper Bounds**

$$\text{(MIP-MP)} \quad \min c(x), \qquad Ax = 0, \quad x_k \in \bigcup_{j \in J_k} \operatorname{conv}(W_{kj}), \quad k \in K$$

$$W_{kj} := \{\hat{x}_{ki} : i \in I_{kj}\} \quad (\hat{x}_{ki}, \hat{y}_{ki}) \in V_k$$

Sequential Approximation (Projection)

1. **Enumerate ε-solutions of MIP-MP (OA)** by Branch & Cut
   (and save valid cuts for warm start)

2. For every ε-OA-solution: **(LB-improvement)**
   a) **project the OA-solution** onto the feasible set **G**
   b) **local refinement of PLEs** in the neighborhood of the projected solution

3. Compute a (or several) **local minimizer(s)** of problem (P) starting from the **ε-OA-solution(s)** **(UB-improvement)**

4. **Local refinement of PLEs** in the neighborhood of the minimizers

Solving the master problems

MIP master problem solver for Step 1:

(in the primary space)

- **Branch&Cut**
- **Many outer points -> active-set strategy:**
  - use a **bundle of points** of the OA which are **active** in LPs of **B&B partitions**
  - use **global cuts** which are computed by **aggregating B&B cuts**

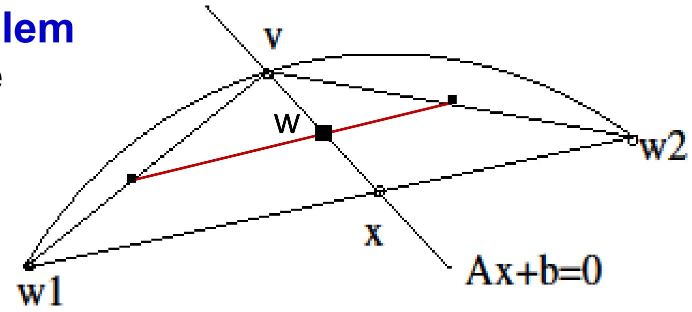
MINLP local search for Step 3:

(in the original space)

- **NLP** or **feasibility pump**
- or **search direction** by solving a **perturbed problem** defined by shifting active outer-points towards the projected OA solution
- **UB-LB gives ε**

w: solution of perturbed OA

v: projected OA solution

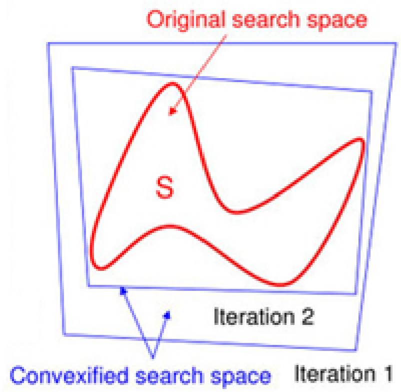


w-x: search direction

Solving the sub-problems

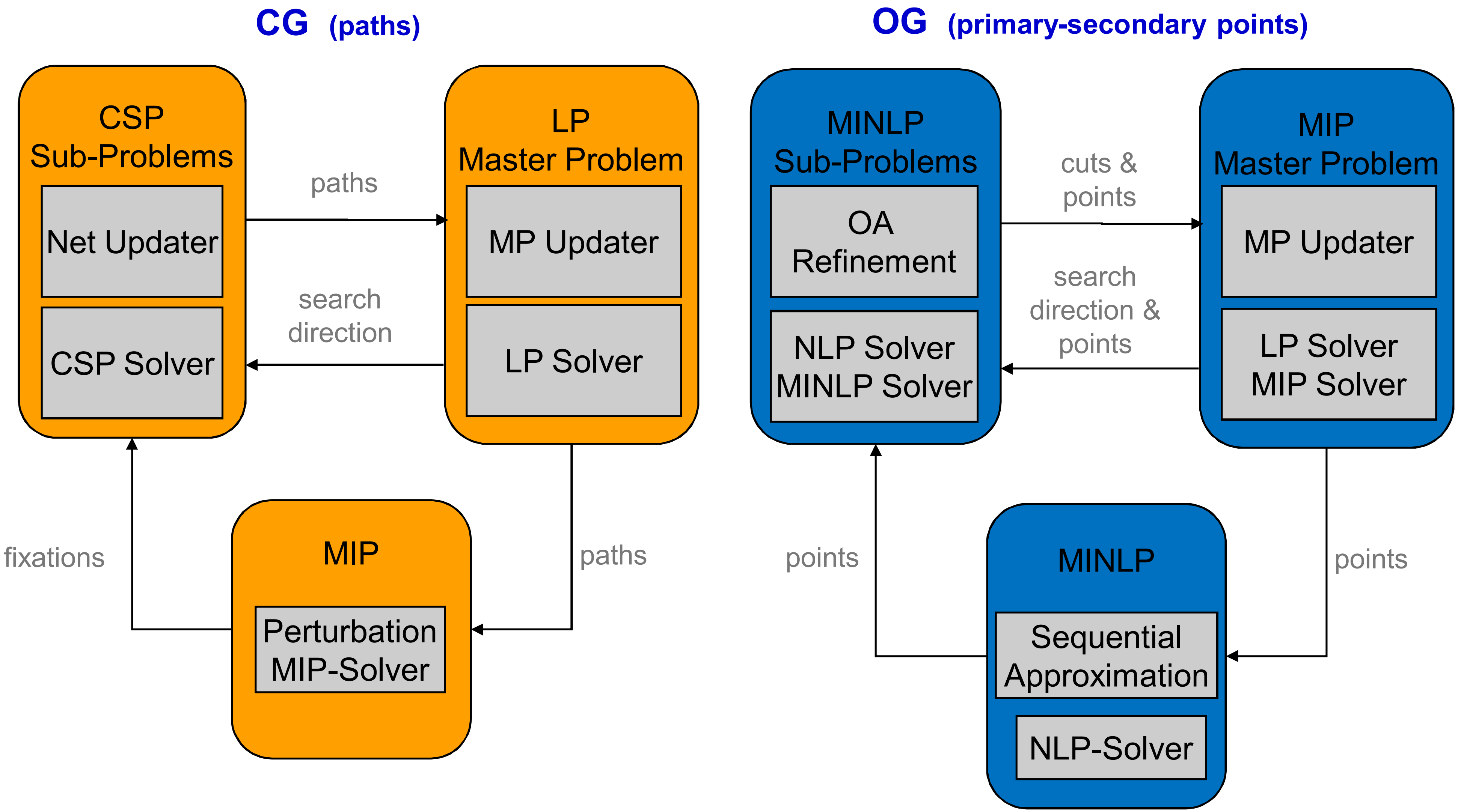
Two MINLP sub-problem solvers for pricing (CG) and projection (Step 3)
(in the primary and secondary space)

- Start with **MINLP pre-solving** where cuts are stored for **warm start**
- **NLP** or **feasibility pump** for computing UB
  - starting from **ε-solutions** of **sub-OA**
- **MINLP-Branch&Cut** and/or **Sequential Approximation**



MINLP pre-solving

CG versus OG


CG (paths)
OG (primary-secondary points)
CSP
Sub-Problems
Net Updater
CSP Solver
paths
search
direction
LP
Master Problem
MP Updater
LP Solver
fixations
MIP
Perturbation
MIP-Solver
paths
MINLP
Sub-Problems
OA
Refinement
NLP Solver
MINLP Solver
cuts &
points
search
direction &
points
MIP
Master Problem
MP Updater
LP Solver
MIP Solver
points
MINLP
Sequential
Approximation
NLP-Solver
points

Agenda

Motivation: Path Based Column Generation

MINLP Outer-Point Generation

Implementation in SCIP

Next Steps

Proposal for the development of a decomposition framework

1. **CG for MIQQPs** using **SCIP/GCG** and **SCIP/MINLP as a pricing-solver**
   - unit test, local search, e.g. using feasibility pump or new perturbation heuristic

2. **Parallel decomposition framework using a MIP master problem**
   - for MIQQP
   - based on GCG

3. **Extension to MINLPs**
   - automatic quasi-separable reformulation by partitioning the sparsity graph
   - point-based OA using PLEs

Agenda

Motivation: Path Based Column Generation

MINLP Outer-Point Generation

Implementation in SCIP

Next Steps

Next Steps

- We presented the OG algorithm with similar properties as CG
- Discussion on **algorithmic details**, e.g. active set strategy, master and sub-solvers,..
- Finishing **internal report** “The Outer-Point Generation Algorithm”
- Implementation of a prototype for a **specific application**
  - e.g. a bilinear problem
- **Applications**
  - engineering optimization (HAW)
  - transport (ZIB)
  - energy conversion (TU-Berlin)
  - ..

Literature

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Thank you!