Cutting Plane Separators in SCIP

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Mathematics for key technologies
MIP

\[
\min \{ c^T x : x \in X^{\text{MIP}} \}, \quad X^{\text{MIP}} := \{ x \in \mathbb{Z}^n \times \mathbb{R}^m : Ax \leq b \}
\]
MIP

\[
\min \{ c^T x : x \in X^{\text{MIP}} \}, \quad X^{\text{MIP}} := \{ x \in \mathbb{Z}^n \times \mathbb{R}^m : Ax \leq b \}
\]

Observation

- If the data are rational, then
  - \( \text{conv}(X^{\text{MIP}}) \) is a rational polyhedron
  - we can formulate the MIP as

\[
\min_{\text{LP}} \{ c^T x : x \in \text{conv}(X^{\text{MIP}}) \}
\]
General Cutting Plane Method

MIP

\[ \min \{ c^T x : x \in X^{MIP} \}, \quad X^{MIP} := \{ x \in \mathbb{Z}^n \times \mathbb{R}^m : Ax \leq b \} \]

Problem (in general)

- Complete linear description of \( \text{conv}(X^{MIP}) \)?
- Number of constraints needed to describe \( \text{conv}(X^{MIP}) \) is extremely large
General Cutting Plane Method

MIP

\[ \min \{ c^T x : x \in X^{MIP} \}, \quad X^{MIP} := \{ x \in \mathbb{Z}^n \times \mathbb{R}^m : Ax \leq b \} \]

Idea

- Construct a polyhedron \( Q \) with
  - \( \text{conv}(X^{MIP}) \subseteq Q \subseteq X^{LP} \)
  - \( \min \{ c^T x : x \in \text{conv}(X^{MIP}) \} = \min \{ c^T x : x \in Q \} \)
General Cutting Plane Method

MIP

\[
\min\{c^T x : x \in X^{MIP}\}, \quad X^{MIP} := \{x \in \mathbb{Z}^n \times \mathbb{R}^m : Ax \leq b\}
\]

Idea

▷ Construct a polyhedron \( Q \) with
  
  ▷ \( \text{conv}(X^{MIP}) \subseteq Q \subseteq X^{LP} \)

  ▷ \( \min\{ c^T x : x \in \text{conv}(X^{MIP}) \} = \min\{ c^T x : x \in Q \} \)

⇔ Start with \( X^{LP} \) and add inequalities which are valid for \( X^{MIP} \) (but violated by the current LP solution)
Main Solving Loop

Domain Propagation

Solve LP

Pricing

Separation

LP Solving

Constraint Enforcement

Domain Propagation
# Two Classes of Cuts

## General cuts
- Complemented mixed integer rounding cuts
- Gomory mixed integer cuts
- Strong Chvátal-Gomory cuts
- Implied bound cuts

## Problem specific cuts
- 0-1 knapsack problem
- 0-1 single node flow problem
- Stable set problem
## Two Classes of Cuts

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### Two Classes of Cuts

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I want to solve general MIPs!
Why do I care about cutting planes for special problems?
## Two Classes of Cuts

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Which plugins of SCIP generate cuts?
Constraint Handlers

- Check a given solution for feasibility w.r.t. all constraints of their type
- Provide linear relaxations of their constraints (in advance or on the fly)
- Provide additional problem specific cuts
Constraint Handlers

- Check a given solution for feasibility w.r.t. all constraints of their type
- Provide linear relaxations of their constraints (in advance or on the fly)
- Provide additional problem specific cuts

Separators

- Provide general cuts
Constraint Handler or Separator?

Type of cuts?

General cuts

Separator

Problem specific cuts

Can your cons be expressed by a "small" nr. of existing cons types?

GMI
Constraint Handler or Separator?

Type of cuts?

- General cuts
- Problem specific cuts

Separator

Can your cons be expressed by a "small" nr. of existing cons types?

- Yes
- No

GMI

Can you represent and process your cons in a **more efficient way**?

- Yes
- No

Constraint handler

LOP
Type of cuts?

- General cuts
  - Separator
    - GMI
  - Can your cons be expressed by a "small" nr. of existing cons types?
    - Yes
      - Can you represent and process your cons in a more efficient way?
        - Yes
          - Constraint handler
        - No
          - Constraint handler
          - LOP
      - No
        - Constraint handler
        - 0-1 SNFP
    - No
      - Constraint handler
      - 0-1 KP
By default, only globally valid cuts are generated.

- FREQ = 0 and SEPAFREQ = 0

Cuts are generated in rounds.

- MAXROUNDSROOT

Cuts are first selected in a separation storage.
Efficacy, i.e., distance of the hyperplane to the LP sol
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- **Orthogonality** with respect to the other cuts
Cut Selection Strategy

- **Efficacy**, i.e., distance of the hyperplane to the LP sol
- **Orthogonality** with respect to the other cuts
- **Parallelism** with respect to the objective function
Cut Selection Strategy

- **Efficacy**, i.e., distance of the hyperplane to the LP solution

- **Orthogonality** with respect to the other cuts

- **Parallelism** with respect to the objective function

⇒ **Select** cuts with largest value of

\[ e_r + w_o \times o_r + w_p \times p_r \]
Cut Selection Strategy

- **Efficacy**, i.e., distance of the hyperplane to the LP solution
  \[ e_r \]

- **Orthogonality** with respect to the other cuts
  \[ o_r \]

- **Parallelism** with respect to the objective function
  \[ p_r \]

⇒ **Select** cuts with largest value of

\[ e_r + w_o \cdot o_r + w_p \cdot p_r \]

**Consequence**

- Cut as deep as possible into the current LP polyhedron
- Select cuts that are pairwise almost orthogonal
- Prefer cuts that are close to being parallel to the objective function
Cut Selection Strategy

- **Efficacy**, i.e., distance of the hyperplane to the LP sol
  \[ e_r \]

- **Orthogonality** with respect to the other cuts
  \[ o_r \]

- **Parallelism** with respect to the objective function
  \[ p_r \]

⇒ **Select** cuts with largest value of
  \[ e_r + w_o \cdot o_r + w_p \cdot p_r \]

Weights of the criteria can be adjusted

- **ORTHOFAC = 1.0**
- **OBJPARALFAC = 0.0001**
Cut Selection Strategy

- **Efficacy**, i.e., distance of the hyperplane to the LP sol
  \( e_r \)

- **Orthogonality** with respect to the other cuts
  \( o_r \)

- **Parallelism** with respect to the objective function
  \( p_r \)

⇒ **Select** cuts with largest value of
\[
e_r + w_o \cdot o_r + w_p \cdot p_r
\]

Other useful parameters

- **MINEFFICACYROOT** = 0.01
- **MAXCUTSROOT** = 2000
Outline

1. Complemented Mixed Integer Rounding Cuts
2. Gomory Mixed Integer and Strong Chvátal-Gomory Cuts
3. Implied Bound Cuts
4. Cuts for the 0-1 Knapsack Problem
5. Cuts for the 0-1 Single Node Flow Problem
6. Cuts for the Stable Set Problem
Complemented Mixed Integer Rounding Cuts

Gomory Mixed Integer and Strong Chvátal-Gomory Cuts

Implied Bound Cuts

Cuts for the 0-1 Knapsack Problem

Cuts for the 0-1 Single Node Flow Problem

Cuts for the Stable Set Problem
Elementary mixed integer set

\[ X := \{(x, s) \in \mathbb{Z} \times \mathbb{R} : x \leq a_0 + s \quad (I) \} \]

\[ s \geq 0 \quad (II) \]
Elementary mixed integer set

\[ X := \{(x, s) \in \mathbb{Z} \times \mathbb{R} : x \leq a_0 + s \text{ (I)} \quad s \geq 0 \text{ (II)}\} \]

Inequalities (I) and (II) do not suffice to describe \( \text{conv}(X) \).

\[ x \leq \lfloor a_0 \rfloor + s - f_{a_0} \]

\( f_{a_0} := a_0 - \lfloor a_0 \rfloor \)
Elementary mixed integer set

\[ \begin{align*}
X := \{ (x, s) &\in \mathbb{Z} \times \mathbb{R} : x \leq a_0 + s \ (I) \\
&\quad \ s \geq 0 \quad (II) \} 
\end{align*} \]

Disjunctive argument

If an inequality is valid for \( X^1 \) and \( X^2 \) it is also valid for \( X^1 \cup X^2 \).

![Diagram](image)
Elementary mixed integer set

\[ X := \{(x, s) \in \mathbb{Z} \times \mathbb{R} : x \leq a_0 + s \ (I) \] 
\[ s \geq 0 \quad (II) \} \]

Here

\[ X^1 := X \cap \{(x, s) : x \geq \lfloor a_0 \rfloor \ (III) \} \]
\[ X^2 := X \cap \{(x, s) : x \leq \lceil a_0 \rceil \ (IV) \} \]
Elementary mixed integer set

\[ X := \{(x, s) \in \mathbb{Z} \times \mathbb{R} : x \leq a_0 + s \ (I) \quad s \geq 0 \ (II)\} \]

Here

\[ X^1 := X \cap \{(x, s) : x \geq \lfloor a_0 \rfloor \ (III)\} \]

\[ X^2 := X \cap \{(x, s) : x \leq \lfloor a_0 \rfloor \ (IV)\} \]

Inequality valid for \( X^1 \) and \( X^2 \)

\[ x \leq \lfloor a_0 \rfloor + \frac{s}{1 - f_{a_0}} \]

\((I) + f_{a_0} (III) \) and \((II) + (1 - f_{a_0}) (IV) \)

\((f_{a_0} := a_0 - \lfloor a_0 \rfloor)\)
Elementary mixed integer set

\[ X := \{(x, s) \in \mathbb{Z} \times \mathbb{R} : x \leq a_0 + s \ (I) \quad s \geq 0 \ (II)\} \]

Here

\[ X^1 := X \cap \{(x, s) : x \geq \lfloor a_0 \rfloor \ (III)\} \]
\[ X^2 := X \cap \{(x, s) : x \leq \lceil a_0 \rceil \ (IV)\} \]

Inequality valid for \( X^1 \cup X^2 = X \)

\[ x \leq \lfloor a_0 \rfloor + \frac{s}{1 - f_{a_0}} \]

MIR inequality

\((f_{a_0} := a_0 - \lfloor a_0 \rfloor)\)
Mixed knapsack set

\[ X^{MK} := \{(x, s) \in \mathbb{Z}_+^n \times \mathbb{R}_+ : \sum_{j \in N} a_j x_j \leq a_0 + s, \quad x_j \leq b_j \text{ for all } j \in N\} \]

- \( N = \{1, \ldots, n\} \)
- \( a_0 \) and \( a_j \) are rational numbers for all \( j \in N \)
- \( b_j \) are nonnegative rational numbers for all \( j \in N \)
**Mixed knapsack set**

\[
X^{MK} := \{(x, s) \in \mathbb{Z}_+^n \times \mathbb{R}_+ : \sum_{j \in N} a_j x_j \leq a_0 + s, \quad x_j \leq b_j \text{ for all } j \in N\}
\]

**MIR function**

\[
F_\alpha : \mathbb{R} \rightarrow \mathbb{R}
\]

\[
d \mapsto F_\alpha(d) := \lfloor d \rfloor + \frac{(f_d - \alpha)^+}{1-\alpha}
\]

- \(0 \leq \alpha < 1\)
- \(f_d := d - \lfloor d \rfloor\)
- \(d^+ := \max\{d, 0\}\)
Mixed knapsack set

\[ X^{MK} := \{ (x, s) \in \mathbb{Z}_+^n \times \mathbb{R}_+ : \sum_{j \in N} a_j x_j \leq a_0 + s, \quad x_j \leq b_j \text{ for all } j \in N \} \]

MIR inequality

\[ \sum_{j \in N} F_{f_{a_0}}(a_j) x_j \leq \lfloor a_0 \rfloor + \frac{s}{1 - f_{a_0}} \]
Mixed knapsack set

\[ X^{MK} := \{(x, s) \in \mathbb{Z}_+^n \times \mathbb{R}_+ : \sum_{j \in N} a_j x_j \leq a_0 + s, \] \[ x_j \leq b_j \text{ for all } j \in N\} \]

MIR inequality

\[ \sum_{j \in N} F_{\alpha_0}(a_j) x_j \leq \lfloor a_0 \rfloor + \frac{s}{1 - f_{\alpha_0}} \]

C-MIR inequality

- Divide cons by \( \delta \in \mathbb{Q}_+ \setminus \{0\} \)
- Complement int vars \( (x_j = b_j - \bar{x}_j) \)
- MIR inequality
\[ \sum_{j \in N} a_j x_j \leq a_0 + s \quad \rightsquigarrow \quad \sum_{j \in N} F_{f_{a_0}}(a_j) x_j \leq \lfloor a_0 \rfloor + \frac{s}{1 - f_{a_0}} \]
\[
\sum_{j \in N} a_j x_j \leq a_0 + s
\]

\[
\sum_{j \in N} F_{f_{a_0}}(a_j)x_j \leq \lfloor a_0 \rfloor + \frac{s}{1 - f_{a_0}}
\]

1 \( x_1 \) + 4 \( x_2 \) \( \leq \frac{11}{2} + s \)

Bounds: \( x_1, x_2 \leq 2 \)
\[
\sum_{j \in N} a_j x_j \leq a_0 + s \\
\sum_{j \in N} F_{f_{a_0}}(a_j)x_j \leq \lfloor a_0 \rfloor + \frac{s}{1 - f_{a_0}}
\]

1\,x_1 + 4\,x_2 \leq \frac{11}{2} + s

Bounds: \( x_1, x_2 \leq 2 \)

For \( \delta = 1 \)
\[ \sum_{j \in N} a_j x_j \leq a_0 + s \]

\[ \sum_{j \in N} F_{f_0}(a_j) x_j \leq \lfloor a_0 \rfloor + \frac{s}{1 - f_{a_0}} \]

For \( \delta = 1 \)

\[ f_{\frac{11}{2}} = \frac{11}{2} - \lfloor \frac{11}{2} \rfloor = \frac{1}{2} \]
\[
\sum_{j \in N} a_j x_j \leq a_0 + s
\]

\[
\sum_{j \in N} F_{f_{a_0}}(a_j) x_j \leq \lfloor a_0 \rfloor + \frac{s}{1 - f_{a_0}}
\]

\[
1 x_1 + 4 x_2 \leq \frac{11}{2} + s
\]

Bounds: \(x_1, x_2 \leq 2\)

For \(\delta = 1\)

\[
\Delta f_{1\frac{11}{2}} = \frac{11}{2} - \lfloor \frac{11}{2} \rfloor = \frac{1}{2}
\]

\[
F_{\frac{1}{2}}(d')
\]

$\delta = 1$

\[
\frac{1}{2}
\]

$1$

$2$

$-1$

$\frac{1}{2}$

$1$

$2$

$d$
\[ \sum_{j \in N} a_j x_j \leq a_0 + s \]

\[ \sum_{j \in N} F_{f(a_j)} x_j \leq \lfloor a_0 \rfloor + \frac{s}{1 - f_{a_0}} \]

\[ 1x_1 + 4x_2 \leq \frac{11}{2} + s \]

Bounds: \( x_1, x_2 \leq 2 \)

For \( \delta = 1 \)

\[ f_{\frac{11}{2}} = \frac{11}{2} - \lfloor \frac{11}{2} \rfloor = \frac{1}{2} \]

\[ F_{\frac{1}{2}}(d) \]

\[ 1x_1 + 4x_2 \leq 5 + 2s \]
\[ \sum_{j \in N} a_j x_j \leq a_0 + s \]

\[ \sum_{j \in N} F_{f_{a_0}}(a_j) x_j \leq \lfloor a_0 \rfloor + \frac{s}{1 - f_{a_0}} \]

1 \( x_1 \) + 4 \( x_2 \) \( \leq \frac{11}{2} + s \)

Bounds: \( x_1, x_2 \leq 2 \)

1 \( x_1 \) + 4 \( x_2 \) \( \leq 5 + 2s \)

For \( \delta = 4 \), \( x_1 = 2 - \bar{x}_1 \)
\[
\sum_{j \in \mathbb{N}} a_j x_j \leq a_0 + s \\
\sum_{j \in \mathbb{N}} F_{f_{a_0}}(a_j) x_j \leq \lfloor a_0 \rfloor + \frac{s}{1 - f_{a_0}}
\]

\[
1x_1 + 4x_2 \leq \frac{11}{2} + s
\\
1x_1 + 4x_2 \leq 5 + 2s
\]

Bounds: \(x_1, x_2 \leq 2\)

For \(\delta = 4\), \(x_1 = 2 - \bar{x}_1\)

\[\frac{1}{4} \bar{x}_1 + x_2 \leq \frac{7}{8} + \frac{1}{4}s\]
\[ \sum_{j \in N} a_j x_j \leq a_0 + s \]
\[ \implies \sum_{j \in N} F_{f_{a_j}}(a_j) x_j \leq \lfloor a_0 \rfloor + \frac{s}{1 - f_{a_0}} \]

\[ 1x_1 + 4x_2 \leq \frac{11}{2} + s \]
\[ \implies 1x_1 + 4x_2 \leq 5 + 2s \]

Bounds: \(x_1, x_2 \leq 2\)

For \(\delta = 4\), \(x_1 = 2 - \bar{x}_1\)

\[ -\frac{1}{4} \bar{x}_1 + x_2 \leq \frac{7}{8} + \frac{1}{4}s \]

\[ f_{\frac{7}{8}} = \frac{7}{8} - \lfloor \frac{7}{8} \rfloor = \frac{7}{8} \]
$$\sum_{j \in N} a_j x_j \leq a_0 + s$$

$$\sum_{j \in N} F_{f_0}(a_j)x_j \leq \lfloor a_0 \rfloor + \frac{s}{1 - f_{a_0}}$$

$$1x_1 + 4x_2 \leq \frac{11}{2} + s$$

$$1x_1 + 4x_2 \leq 5 + 2s$$

Bounds: $x_1, x_2 \leq 2$

For $\delta = 4$, $x_1 = 2 - \bar{x}_1$

$$-\frac{1}{4}\bar{x}_1 + x_2 \leq \frac{7}{8} + \frac{1}{4}s$$

$$f_{\frac{7}{8}} = \frac{7}{8} - \lfloor \frac{7}{8} \rfloor = \frac{7}{8}$$
\[ \sum_{j \in N} a_j x_j \leq a_0 + s \]

\[ \sum_{j \in N} F_{f_{a_0}}(a_j) x_j \leq \lfloor a_0 \rfloor + \frac{s}{1 - f_{a_0}} \]

1\(x_1 + 4x_2 \leq \frac{11}{2} + s\)

Bounds: \(x_1, x_2 \leq 2\)

For \(\delta = 4\), \(x_1 = 2 - \bar{x}_1\)

\[ -\frac{1}{4} \bar{x}_1 + x_2 \leq \frac{7}{8} + \frac{1}{4} s \]

\[ f_{\frac{7}{8}} = \frac{7}{8} - \lfloor \frac{7}{8} \rfloor = \frac{7}{8} \]
\[
\sum_{j \in N} a_j x_j \leq a_0 + s
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\[
\sum_{j \in N} F_{f_{a_0}}(a_j) x_j \leq \lfloor a_0 \rfloor + \frac{s}{1 - f_{a_0}}
\]

\[
1 x_1 + 4 x_2 \leq \frac{11}{2} + s
\]

Bounds: \( x_1, x_2 \leq 2 \)

\[
1 x_1 + 4 x_2 \leq 5 + 2s
\]

\[
1 x_1 + 1 x_2 \leq 2 + 2s
\]

For \( \delta = 4 \), \( x_1 = 2 - \bar{x}_1 \)

\[\begin{align*}
& -\frac{1}{4} \bar{x}_1 + x_2 \leq \frac{7}{8} + \frac{1}{4} s \\
& f_{\frac{7}{8}} = \frac{7}{8} - \lfloor \frac{7}{8} \rfloor = \frac{7}{8}
\end{align*}\]
Mixed knapsack constraint (relaxation of MIP)

Cut generation heuristic

▷ Choose $\delta \in \mathbb{Q}_+ \setminus \{0\}$

▷ Choose $U \subseteq N$

⚠️ Violated c-MIR inequality
Outline of the Separation Algorithm

Mixed integer constraints (MIP)

Aggregation heuristic

Aggregated mixed integer constraint

Bound substitution heuristic

Mixed knapsack constraint (relaxation of MIP)

Cut generation heuristic

> Choose $\delta \in \mathbb{Q}_+ \setminus \{0\}$

> Choose $U \subseteq N$

Violated c-MIR inequality
Efficiency of the Separation Algorithm

Aggregation heuristic

- Prefer constraints with
  - large LP solution value of the dual variable
  - small density
  - small slack

and which

- have not been involved in an aggregation before
Efficiency of the Separation Algorithm

Aggregation heuristic

- Prefer constraints with
  - large LP solution value of the dual variable
  - small density
  - small slack

  and which

  - have not been involved in an aggregation before

Reducing the separation time

- Limit the number of starting constraints

  - Use the same criterion as in the aggregation heuristic
  - $\text{MAXFAILS} = 150$

- In addition

  - $\text{MAXCUTS} = 100$ and $\text{MAXROUNDS} = 50$
1. Complemented Mixed Integer Rounding Cuts
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6. Cuts for the Stable Set Problem
### Gomory mixed integer (GMI) inequalities

- Equivalent to MIR inequalities
- Pure integer case: dominate CG inequalities
### Gomory mixed integer (GMI) inequalities

- Equivalent to MIR inequalities
- Pure integer case: dominate CG inequalities

### Strong Chvátal-Gomory (CG) inequalities

- No dominance relation to MIR inequalities
- Pure integer case: dominate CG inequalities
Mixed integer constraints (MIP)

\[ \Downarrow \quad \text{Aggregation heuristic} \]

Aggregated mixed integer constraint

\[ \Downarrow \quad \text{Bound substitution heuristic} \]

Mixed knapsack constraint (relaxation of MIP)

\[ \Downarrow \quad \text{Cut generation heuristic} \]

Choose \( \delta \in \mathbb{Q}_+ \setminus \{0\} \) and \( U \subseteq N \)

Violated c-MIR inequality
Mixed integer constraints (MIP)

**Aggregation heuristic**
- Weights of $A_B^{-1}$

Row of the simplex tableau – for integer var with fractional LP val

**Bound substitution heuristic**
- Nearly turned off

Mixed knapsack constraint (relaxation of MIP)

**Cut generation heuristic**
- $\delta = 1$ and $U$ is nearly empty

**Violated c-MIR inequality**
Mixed integer constraints (MIP)

**Aggregation heuristic**
- Weights of $A_B^{-1}$

**Bound substitution heuristic**
- Nearly turned off

Row of the simplex tableau – no real vars with negative coeffs

**Cut generation heuristic**
- $\delta = 1$ and $U$ is nearly empty
- Apply strong CG function

Mixed knapsack constraint (relaxation of MIP)

Violated strong CG inequality
1. Complemented Mixed Integer Rounding Cuts
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6. Cuts for the Stable Set Problem
Extended in preprocessing and probing:

**Implication graph**

Represents *logical implications* of the form

\[
\begin{align*}
x = v & \rightarrow y \leq b \\
x = v & \rightarrow y \geq b.
\end{align*}
\]

- $x$ is a binary variable and $v \in \{0, 1\}$
- $y \in [l, u]$ is an arbitrary variable
Extended in preprocessing and probing:

**Implication graph**

Represents *logical implications* of the form

\[
\begin{align*}
  x = v \rightarrow y & \leq b \\
  x = v \rightarrow y & \geq b.
\end{align*}
\]

- $x$ is a binary variable and $v \in \{0, 1\}$
- $y \in [l, u]$ is an arbitrary variable

*Used to separate implied bound cuts* of the form

- $y \leq cx + d$ (variable upper bounds)
- $y \geq cx + d$ (variable lower bounds).
Logical implication

$x = 0 \rightarrow y \leq b$

Implied bound cut

$y \leq (u - b)x + b = \begin{cases} b & : x = 0 \\ u & : x = 1 \end{cases}$
Logical implication

\[ x = 0 \Rightarrow y \leq b \]

\[ x = 0 \Rightarrow y \geq b \]

\[ y \leq (u - b)x + b = \begin{cases} b : x = 0 \\ u : x = 1 \end{cases} \]

\[ y \geq (l - b)x + b = \begin{cases} b : x = 0 \\ l : x = 1 \end{cases} \]
Logical implication

\[ x = 0 \rightarrow y \leq b \]

\[ x = 0 \rightarrow y \geq b \]

\[ x = 1 \rightarrow y \leq b \]

Implied bound cut

\[ y \leq (u - b)x + b = \begin{cases} b & : x = 0 \\ u & : x = 1 \end{cases} \]

\[ y \geq (l - b)x + b = \begin{cases} b & : x = 0 \\ l & : x = 1 \end{cases} \]

\[ y \leq (b - u)x + u = \begin{cases} u & : x = 0 \\ b & : x = 1 \end{cases} \]
Implied Bound Cuts

Logical implication

\[ x = 0 \rightarrow y \leq b \]

\[ x = 0 \rightarrow y \geq b \]

\[ x = 1 \rightarrow y \leq b \]

\[ x = 1 \rightarrow y \geq b \]

Implied bound cut

\[ y \leq (u - b)x + b = \begin{cases} b & : x = 0 \\ u & : x = 1 \end{cases} \]

\[ y \geq (l - b)x + b = \begin{cases} b & : x = 0 \\ l & : x = 1 \end{cases} \]

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\[ y \geq (b - l)x + l = \begin{cases} l & : x = 0 \\ b & : x = 1 \end{cases} \]
1. Complemented Mixed Integer Rounding Cuts
2. Gomory Mixed Integer and Strong Chvátal-Gomory Cuts
3. Implied Bound Cuts
4. Cuts for the 0-1 Knapsack Problem
5. Cuts for the 0-1 Single Node Flow Problem
6. Cuts for the Stable Set Problem
0-1 knapsack polytope

\[
\text{conv}(X^{BK}),
\]

\[
X^{BK} := \{x \in \{0,1\}^n : \sum_{j \in N} a_j x_j \leq a_0 \}
\]

- \(N = \{1, \ldots, n\}\)
- \(a_0\) and \(a_j\) are integers for all \(j \in N\)
- \(0 \leq a_j \leq a_0\) for all \(j \in N\)
**Cuts for the 0-1 Knapsack Problem**

### 0-1 Knapsack Polytope

<table>
<thead>
<tr>
<th>conv($X^{BK}$),</th>
<th>$X^{BK} := { x \in {0, 1}^n : \sum_{j \in N} a_j x_j \leq a_0 } $</th>
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</table>

### Minimal Cover:

<table>
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<tr>
<th>$C \subseteq N$</th>
</tr>
</thead>
</table>

with

- $\sum_{j \in C} a_j > a_0$
- $\sum_{j \in C \setminus \{i\}} a_j \leq a_0 \ \forall i \in C$
Cuts for the 0-1 Knapsack Problem

### 0-1 knapsack polytope

\[ \text{conv}(X^{BK}), \quad X^{BK} := \{ x \in \{0, 1\}^n : \sum_{j \in N} a_j x_j \leq a_0 \} \]

### Minimal cover:

\[ C \subseteq N \]

with

- \( \sum_{j \in C} a_j > a_0 \)
- \( \sum_{j \in C \setminus \{i\}} a_j \leq a_0 \ \forall \ i \in C \)

- Minimal cover: \( C = \{2, 3, 4\} \)

- \( 5x_1 + 6x_2 + 2x_3 + 2x_4 \leq 8 \)
0-1 knapsack polytope

\[ \text{conv}(X^{BK}), \quad X^{BK} := \{ x \in \{0, 1\}^n : \sum_{j \in N} a_j x_j \leq a_0 \} \]

Minimal cover inequality:

\[ \sum_{j \in C} x_j \leq |C| - 1 \]

defines a facet of \( \text{conv}(X^{BK} \cap \{ x \in \{0, 1\}^n : x_j = 0 \ \forall j \in N \setminus C \}) \)

- \( 5x_1 + 6x_2 + 2x_3 + 2x_4 \leq 8 \)
- Minimal cover: \( C = \{2, 3, 4\} \)
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$$\text{conv}(X^{BK}),$$

$$X^{BK} := \{ x \in \{0, 1\}^n : \sum_{j \in N} a_j x_j \leq a_0 \}$$

Minimal cover inequality:

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- \( 5x_1 + 6x_2 + 2x_3 + 2x_4 \leq 8 \)
- Minimal cover: \( C = \{2, 3, 4\} \)
- \( x_2 + x_3 + x_4 \leq 2 \)
defines a facet of \( \text{conv}(X^{BK} \cap \{ x \in \{0, 1\}^4 : x_1 = 0 \}) \)
0-1 knapsack polytope

\[ \text{conv}(X^{BK}), \quad X^{BK} := \{ x \in \{0, 1\}^n : \sum_{j \in N} a_j x_j \leq a_0 \} \]

Sequential up-lifting:

- Used to strengthen minimal cover inequalities

- 5x_1 + 6x_2 + 2x_3 + 2x_4 \leq 8
- Minimal cover: \( C = \{2, 3, 4\} \)
- \( x_2 + x_3 + x_4 \leq 2 \) defines a facet of \( \text{conv}(X^{BK} \cap \{ x \in \{0, 1\}^4 : x_1 = 0 \}) \)
$X^{BK} = \{ x \in \{0, 1\}^4 : 5x_1 + 6x_2 + 2x_3 + 2x_4 \leq 8 \},$

$C = \{2, 3, 4\}$

$\sum_{j \in C} x_j \leq 2 \quad \text{valid for } X^0 := X^{BK} \cap \{ x \in \{0, 1\}^4 : x_1 = 0 \}$

(I) $\sum_{j \in C} x_j + \alpha_1 x_1 \leq 2 \quad \text{valid for } X^1 := X^{BK}$
\( X^{BK} = \{ x \in \{0, 1\}^4 : 5x_1 + 6x_2 + 2x_3 + 2x_4 \leq 8 \} \),

\( C = \{2, 3, 4\} \)

\[
\sum_{j \in C} x_j \leq 2 \quad \text{valid for } X^0 := X^{BK} \cap \{ x \in \{0, 1\}^4 : x_1 = 0 \}
\]

(1) \[
\sum_{j \in C} x_j + \alpha_1 x_1 \leq 2 \quad \text{valid for } X^1 := X^{BK}
\]

**Case 1:** Inequality (1) is valid for \( X^1 \cap \{ x \in \{0, 1\}^4 : x_1 = 0 \} \)

\[
\Leftrightarrow \sum_{j \in C} x_j + \alpha_1 \cdot 0 \leq 2 \text{ is valid for } X^0
\]
\[ X^{BK} = \{ x \in \{0,1\}^4 : 5x_1 + 6x_2 + 2x_3 + 2x_4 \leq 8 \}, \]
\[ C = \{ 2, 3, 4 \} \]

\[
\sum_{j \in C} x_j \leq 2 \quad \text{valid for } X^0 := X^{BK} \cap \{ x \in \{0,1\}^4 : x_1 = 0 \}
\]

(I) \[
\sum_{j \in C} x_j + \alpha_1 x_1 \leq 2 \quad \text{valid for } X^1 := X^{BK}
\]

**Case 1:** Inequality (I) is valid for \( X^1 \cap \{ x \in \{0,1\}^4 : x_1 = 0 \} \)

\[
\Leftrightarrow \sum_{j \in C} x_j + \alpha_1 \cdot 0 \leq 2 \quad \text{is valid for } X^0
\]

\[
\Leftrightarrow \alpha_1 \in [-\infty, \infty]
\]
$X^{BK} = \{ x \in \{0, 1\}^4 : 5x_1 + 6x_2 + 2x_3 + 2x_4 \leq 8 \}$,

$C = \{2, 3, 4\}$

$$\sum_{j \in C} x_j \leq 2 \quad \text{valid for } X^0 := X^{BK} \cap \{ x \in \{0, 1\}^4 : x_1 = 0 \}$$

(I) $$\sum_{j \in C} x_j + \alpha_1 x_1 \leq 2 \quad \text{valid for } X^1 := X^{BK}$$

Case 2: Inequality (I) is valid for $X^1 \cap \{ x \in \{0, 1\}^4 : x_1 = 1 \}$

$$\Leftrightarrow \sum_{j \in C} x_j + \alpha_1 \cdot 1 \leq 2 \text{ is valid for all } x \text{ with } 6x_2 + 2x_3 + 2x_4 \leq 8 - 5$$
\( X^{BK} = \{ x \in \{0, 1\}^4 : 5x_1 + 6x_2 + 2x_3 + 2x_4 \leq 8 \} \),
\( C = \{2, 3, 4\} \)
\[
\sum_{j \in C} x_j \leq 2 \quad \text{valid for} \quad X^0 := X^{BK} \cap \{ x \in \{0, 1\}^4 : x_1 = 0 \}
\]
\( (I) \quad \sum_{j \in C} x_j + \alpha_1 x_1 \leq 2 \quad \text{valid for} \quad X^1 := X^{BK} \)

**Case 2:** Inequality (I) is valid for \( X^1 \cap \{ x \in \{0, 1\}^4 : x_1 = 1 \} \)

\[
\Leftrightarrow \sum_{j \in C} x_j + \alpha_1 \cdot 1 \leq 2 \quad \text{is valid for all} \quad x \text{ with } 6x_2 + 2x_3 + 2x_4 \leq 8 - 5
\]
\[
\Leftrightarrow \max \{ \sum_{j \in C} x_j : 6x_2 + 2x_3 + 2x_4 \leq 8 - 5, \ x \in \{0, 1\}^4 \} + \alpha_1 \cdot 1 \leq 2
\]
\[ X^{BK} = \{ x \in \{0, 1\}^4 : 5x_1 + 6x_2 + 2x_3 + 2x_4 \leq 8 \}, \]
\[ C = \{2, 3, 4\} \]
\[ \sum_{j \in C} x_j \leq 2 \quad \text{valid for } X^0 := X^{BK} \cap \{ x \in \{0, 1\}^4 : x_1 = 0 \} \]
\[ (I) \sum_{j \in C} x_j + \alpha_1 x_1 \leq 2 \quad \text{valid for } X^1 := X^{BK} \]

**Case 2:** Inequality (I) is valid for \( X^1 \cap \{ x \in \{0, 1\}^4 : x_1 = 1 \} \)

\[ \iff \sum_{j \in C} x_j + \alpha_1 \cdot 1 \leq 2 \text{ is valid for all } x \text{ with } 6x_2 + 2x_3 + 2x_4 \leq 8 - 5 \]
\[ \iff \max\{ \sum_{j \in C} x_j : 6x_2 + 2x_3 + 2x_4 \leq 8 - 5, \ x \in \{0, 1\}^4 \} + \alpha_1 \cdot 1 \leq 2 \]
\[ \iff \alpha_1 \leq 2 - 1 = 1 \]
\( X^{BK} = \{ x \in \{0, 1\}^4 : 5x_1 + 6x_2 + 2x_3 + 2x_4 \leq 8 \} \),

\( C = \{2, 3, 4\} \)

\[
\sum_{j \in C} x_j \leq 2 \quad \text{valid for } X^0 := X^{BK} \cap \{ x \in \{0, 1\}^4 : x_1 = 0 \}
\]

(\( I \)) \[
\sum_{j \in C} x_j + \alpha_1 x_1 \leq 2 \quad \text{valid for } X^1 := X^{BK}
\]

**Result:** Inequality (I) is valid for \( X^1 \) for \( \alpha_1 \leq 1 \)
(\(j_1, \ldots, j_t\)) lifting sequence of the variables in \(N \setminus C\)

\[X^i := X^{BK} \cap \{x \in \{0, 1\}^n : x_{j_{i+1}} = \ldots = x_{j_t} = 0\}\]

\[
\begin{align*}
\sum_{j \in C} x_j & \leq |C| - 1 \quad \text{valid for } X^0 \\
\sum_{j \in C} x_j + \alpha_{j_1} x_{j_1} & \leq |C| - 1 \quad \text{valid for } X^1 \\
\vdots \\
\sum_{j \in C} x_j + \sum_{k=1}^t \alpha_{j_k} x_{j_k} & \leq |C| - 1 \quad \text{valid for } X^t = X^{BK}
\end{align*}
\]
$(j_1, \ldots, j_t)$ lifting sequence of the variables in $N \setminus C$

$X^i := X^{BK} \cap \{x \in \{0, 1\}^n : x_{j_{i+1}} = \ldots = x_{j_t} = 0\}$

\[
\sum_{j \in C} x_j \leq |C| - 1 \quad \text{valid for } X^0
\]

\[
\sum_{j \in C} x_j + \alpha_{j_i} x_{j_i} \leq |C| - 1 \quad \text{valid for } X^1
\]

\[\vdots\]

\[
\sum_{j \in C} x_j + \sum_{k=1}^{t} \alpha_{j_k} x_{j_k} \leq |C| - 1 \quad \text{valid for } X^t = X^{BK}
\]

Different lifting sequences may lead to different inequalities!

Use sequential up- and down-lifting!
Theorem

If \( C \subseteq N \) is a minimal cover for \( X^{BK} \) and \((C_1, C_2)\) is any partition of \( C \) with \( C_1 \neq \emptyset \), then inequality

\[
\sum_{j \in C_1} x_j \leq |C_1| - 1
\]

defines a facet of

\[
\text{conv}( X^{BK} \cap \{ x \in \{0, 1\}^n : x_j = 0 \text{ for all } j \in N \setminus C, x_j = 1 \text{ for all } j \in C_2 \}).
\]
Sequential Up- and Down-lifting

**Theorem**

If $C \subseteq N$ is a minimal cover for $X^B_K$ and $(C_1, C_2)$ is any partition of $C$ with $C_1 \neq \emptyset$, then inequality

$$\sum_{j \in C_1} x_j \leq |C_1| - 1$$

defines a facet of

$$\text{conv}( X^B_K \cap \{ x \in \{0, 1\}^n : x_j = 0 \text{ for all } j \in N \setminus C, x_j = 1 \text{ for all } j \in C_2 \} ).$$

- **Up-lifting:** variables in $N \setminus C$
- **Down-lifting:** variables in $C_2$
<table>
<thead>
<tr>
<th>Step 1 (Initial cover)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>▶ Determine an initial cover ( C ) for ( X^{BK} )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Step 2 (Minimal cover and partition)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>▶ Make the initial cover minimal by removing vars from ( C )</td>
<td></td>
</tr>
<tr>
<td>▶ Find a partition ((C_1, C_2)) of ( C ) with ( C_1 \neq \emptyset )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Step 3 (Lifting)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>▶ Determine a lifting sequence of the variables in ( N \setminus C_1 )</td>
<td></td>
</tr>
<tr>
<td>▶ Lift the inequality ( \sum_{j \in C_1} x_j \leq</td>
<td>C_1</td>
</tr>
</tbody>
</table>
0-1 single node flow set

\[ X^{SNF} := \{ (x, y) \in \{0, 1\}^n \times \mathbb{R}_+^n : \sum_{j \in N_1} y_j - \sum_{j \in N_2} y_j \leq b, \quad y_j \leq u_j x_j \text{ for all } j \in N \} \]

- \( N = \{1, \ldots, n\} \)
- \((N_1, N_2)\) is a partition of \( N \)
- \( b \) is a rational number
- \( u_j \) are nonnegative rational numbers for all \( j \in N \)
Cuts for the 0-1 Single Node Flow Problem

0-1 single node flow set

\[ \mathcal{X}^{\text{SNF}} := \{(x, y) \in \{0, 1\}^n \times \mathbb{R}_+^n : \sum_{j \in N_1} y_j - \sum_{j \in N_2} y_j \leq b, \]
\[ y_j \leq u_j x_j \text{ for all } j \in N \} \]

- **External demand** \( b \)
- **Inflow arcs** \( N_1 \)
- **Outflow arcs** \( N_2 \)
- **Capacities**
  - \( u_j \), if \( j \) is open \((x_j = 1)\)
  - \( 0 \), if \( j \) is closed \((x_j = 0)\)
- **Flow conservation constraint**
  - inflow – outflow \( \leq \) demand
0-1 single node flow set

\[ \mathcal{X}^{SNF} := \{(x, y) \in \{0, 1\}^n \times \mathbb{R}_+^n : \sum_{j \in N_1} y_j - \sum_{j \in N_2} y_j \leq b, \]
\[ y_j \leq u_j x_j \text{ for all } j \in N \} \]

Flow cover:

\[ (C_1, C_2) \]

with

- \( C_1 \subseteq N_1 \) and \( C_2 \subseteq N_2 \)
- \( \sum_{j \in C_1} u_j - \sum_{j \in C_2} u_j = b + \lambda \)
- \( \lambda > 0 \)
0-1 single node flow set

\[ \mathcal{X}^{SNF} := \{(x, y) \in \{0, 1\}^n \times \mathbb{R}_+^n : \sum_{j \in N_1} y_j - \sum_{j \in N_2} y_j \leq b, \]
\[ y_j \leq u_j x_j \text{ for all } j \in N\}\}

Flow cover:

\[ (C_1, C_2) \]

with

\[ C_1 \subseteq N_1 \text{ and } C_2 \subseteq N_2 \]
\[ \sum_{j \in C_1} u_j - \sum_{j \in C_2} u_j = b + \lambda \]
\[ \lambda > 0 \]

\[ (C_1, C_2) = (\{1, 2\}, \{4, 5\}) \]
\[ 6 - 2 = 2 + 2 \]
Cuts for the 0-1 Single Node Flow Problem

0-1 single node flow set

\[ X^{SNF} := \{(x, y) \in \{0, 1\}^n \times \mathbb{R}^n_+ : \sum_{j \in N_1} y_j - \sum_{j \in N_2} y_j \leq b, \]

\[ y_j \leq u_j x_j \text{ for all } j \in N \} \]

Flow cover inequalities:

- **SGFCI** dominated by
  - LSGFCI
  - part. **c-MIR inequ.** for part. mixed knapsack relaxation

- **EGFCI** dominated by
  - LFCI
  - part. **c-MIR inequ.** for part. mixed knapsack relaxation

\[(C_1, C_2) = (\{1, 2\}, \{4, 5\})\]

\[6 - 2 = 2 + 2\]
Outline of the Separation Algorithm

Mixed integer constraints (MIP)

\[ \Downarrow \quad \text{Aggregation heuristic} \]

Aggregated mixed integer constraint

\[ \Downarrow \quad \text{Bound substitution heuristic} \]

Mixed knapsack constraint (relaxation of MIP)

\[ \Downarrow \quad \text{Cut generation heuristic} \]

\[ \triangleright \text{Choose } \delta \in \mathbb{Q}_+ \setminus \{0\} \text{ and } U \subseteq N \]

Violated c-MIR inequality
Outline of the Separation Algorithm

Mixed integer constraints (MIP)

- Transformation

0-1 single node flow constraint

- Bound substitution heuristic
  - $C_1, L_1 \subseteq N_1 \setminus C_1$: variable ub
  - $C_2, L_2 \subseteq N_2 \setminus C_2$: variable ub

Mixed knapsack constraint (relaxation of MIP)

- Cut generation heuristic
  - $\delta > \lambda \ (\rightarrow \text{dom. SGFCI and EGFCI})$
  - $U = C_1 \cup C_2$

Violated c-MIR inequality
Efficiency of the Separation Algorithm

Cut generation heuristic

- Extend the candidate set for the value of $\delta$
  (based on coefficients in 0-1 single node flow constraint)

Extension

- Separate c-MIR inequalities based on flow packs in addition

Reducing the separation time

- Use a similar strategy as in the separator for the c-MIR cut
1. Complemented Mixed Integer Rounding Cuts
2. Gomory Mixed Integer and Strong Chvátal-Gomory Cuts
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5. Cuts for the 0-1 Single Node Flow Problem
6. Cuts for the Stable Set Problem
“Conflict graph”

\[ G = (V, E) \]

- \( V \) contains a node for each \textbf{binary variable} \( x_i \) and its complement \( \bar{x}_i \)
- \( (x_i, x_j) \in E \iff \) In any feas sol, \( x_i \) and \( x_j \) \textbf{cannot be 1} at the same time
“Conflict graph”

\[ G = (V, E) \]
Cuts for the Stable Set Problem

“Conflict graph”

\[ G = (V, E) \]

Stable set:

\[ S \subseteq V \]

with

\[ u, v \in S \Rightarrow (u, v) \notin E \]

Relaxation of the MIP:

\[ \Rightarrow \text{Stable set polytope is a relaxation of the feas region} \]
"Conflict graph"

\[
G = (V, E)
\]

Clique:

\[
C \subseteq V
\]

with

\[
\triangleright \quad u, v \in S \implies (u, v) \in E
\]

Clique inequality:

\[
\sum_{j \in C} x_j \leq 1
\]

\[
x_1 + x_2 + \bar{x}_3 \leq 1
\]
“Conflict graph”

\[ G = (V, E) \]

**Separation algo:**

- Branch-and-bound algo for the maximum weighted clique problem

\[ x_1 + x_2 + \bar{x}_3 \leq 1 \]
### Two Classes of Cuts

<table>
<thead>
<tr>
<th>General cuts</th>
<th>Problem specific cuts</th>
</tr>
</thead>
<tbody>
<tr>
<td>▶ Complemented mixed integer rounding cuts</td>
<td>▶ 0-1 knapsack problem</td>
</tr>
<tr>
<td>▶ Gomory mixed integer cuts</td>
<td>▶ 0-1 single node flow problem</td>
</tr>
<tr>
<td>▶ Strong Chvátal-Gomory cuts</td>
<td>▶ Stable set problem</td>
</tr>
<tr>
<td>▶ Implied bound cuts</td>
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</tr>
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Cutting Plane Separators in SCIP

Kati Wolter
Zuse Institute Berlin

DFG Research Center MATHEON
Mathematics for key technologies